向量分析

第一章 向量代數

I. 向量的基本性質

在卡式坐標系統(Cartesian coordinate system)中，定義 **i**、**j** 及 **k** 分別為指向 x、y 及 z 軸正向的單位向量，任何向量 **v** 在這三個軸的投影分別為 v_1、v_2及v_3，**v** 可表為

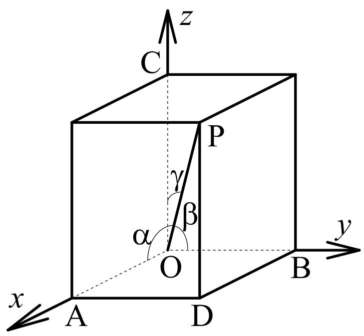
$$\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k} \text{ or } \quad \mathbf{v} = [v_1, v_2, v_3]$$

v_1、v_2及v_3稱為 **v** 在 x、y 及 z 方向的分量。將 **v** 移到它的始點與原點重合，則由 x、y 及 z 軸的正向，量到 **v** 的角度分別以α、β、及γ表示，稱為方向角，則

$$v_1 = v\cos\alpha,\ v_2 = v\cos\beta,\ v_3 = v\cos\gamma,$$

其中$v = |\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$ 為 **v** 的大小，而cosα、cosβ、及cosγ 稱為 **v** 的方向餘弦，明顯地，方向餘弦滿足

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$



且將 **v** 除以它本身的長度即得到一個單位向量，因此任何單位向量皆可表式為$\cos\alpha\mathbf{i} + \cos\beta\mathbf{j} + \cos\gamma\mathbf{k}$。

[向量加法]若 $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k},\ \mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$，則

$$\mathbf{a} + \mathbf{b} = (a_1 + b_1)\mathbf{i} + (a_2 + b_2)\mathbf{j} + (a_3 + b_3)\mathbf{k} = \mathbf{b} + \mathbf{a}$$

$$c\mathbf{a} = ca_1\mathbf{i} + ca_2\mathbf{j} + ca_3\mathbf{k}$$

在平面上一對不平行的向量形成該平面的基底，也就是說，該平面上任何向量都可以唯一表示為此基底的線性組合；同理，三度空間由三個不共面的向量當基底。

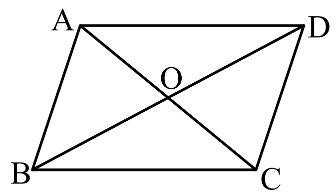
Ex. 1

Find the vector of length 6 in the direction of $\mathbf{u}=\mathbf{i}+2\mathbf{j}-2\mathbf{k}$. [99中山機電III 1 (c)]

[解]$6\dfrac{\mathbf{u}}{|\mathbf{u}|}=6\dfrac{\mathbf{i}+2\mathbf{j}-2\mathbf{k}}{\sqrt{1^2+2^2+(-2)^2}}=6\dfrac{\mathbf{i}+2\mathbf{j}-2\mathbf{k}}{3}=2\mathbf{i}+4\mathbf{j}-4\mathbf{k}$

Ex. 2

Prove that the diagonals of a parallelogram bisect each other.



[解]令 $\overrightarrow{BC}=\mathbf{a}$，$\overrightarrow{BA}=\mathbf{b}$

$\overrightarrow{BO}=m\overrightarrow{BD}=m(\overrightarrow{BC}+\overrightarrow{CD})=m(\overrightarrow{BC}+\overrightarrow{BA})=m(\mathbf{a}+\mathbf{b})$，且

$\overrightarrow{BO}=\overrightarrow{BC}+\overrightarrow{CO}=\overrightarrow{BC}+n\overrightarrow{CA}=\overrightarrow{BC}+n(\overrightarrow{BA}-\overrightarrow{BC})$

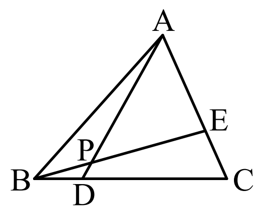
$=(1-n)\overrightarrow{BC}+n\overrightarrow{BA}=(1-n)\mathbf{a}+n\mathbf{b}$

得到 $m(\mathbf{a}+\mathbf{b})=(1-n)\mathbf{a}+n\mathbf{b}\Rightarrow(m+n-1)\mathbf{a}+(m-n)\mathbf{b}=0$

因為 $\mathbf{a}$ 與 $\mathbf{b}$ 為獨立向量，因此 $m+n-1=0$ 且 $m-n=0$，得 $m=n=1/2$，知 O 點為兩對角線的中點，即兩對角線互相平分

Ex. 3

If $\overline{CD}=3\overline{BD}$, $\overline{AE}=2\overline{CE}$, Find $\overline{AP}:\overline{PD}$.



[解]$\overrightarrow{AP}=t\overrightarrow{AD}=t(\overrightarrow{AB}+\overrightarrow{BD})=t(\overrightarrow{AB}+\dfrac{1}{4}\overrightarrow{BC})=t[\overrightarrow{AB}+\dfrac{1}{4}(\overrightarrow{AC}-\overrightarrow{AB})]$

$=t(\dfrac{3}{4}\overrightarrow{AB}+\dfrac{1}{4}\overrightarrow{AC})=\dfrac{3t}{4}\overrightarrow{AB}+\dfrac{t}{4}\overrightarrow{AC}$

$\overrightarrow{AP}=\overrightarrow{AB}+\overrightarrow{BP}=\overrightarrow{AB}+s\overrightarrow{BE}=\overrightarrow{AB}+s(\overrightarrow{AE}-\overrightarrow{AB})$

$=(1-s)\overrightarrow{AB}+\dfrac{2s}{3}\overrightarrow{AC}$

$$\begin{cases}\dfrac{3t}{4}=1-s\\ \dfrac{t}{4}=\dfrac{2s}{3}\end{cases}\Rightarrow t=\dfrac{8}{9}\Rightarrow\overline{AP}:\overline{PD}=8:1$$

Ex. 4

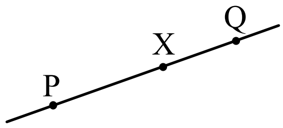
Find the straight line which passes through the two points P(x_1, y_1, z_1), Q(x_2, y_2, z_2).

[解]設直線上任一點為X(x, y, z)，則

$$\overrightarrow{\mathrm{PX}} = t\overrightarrow{\mathrm{PQ}} \Rightarrow [(x-x_1)\mathbf{i}+(y-y_1)\mathbf{j}+(z-z_1)\mathbf{k}] = t[(x_2-x_1)\mathbf{i}+(y_2-y_1)\mathbf{j}+(z_2-z_1)\mathbf{k}]$$

$x-x_1 = t(x_2-x_1)$，$y-y_1 = t(y_2-y_1)$，$z-z_1 = t(z_2-z_1)$

$\dfrac{x-x_1}{x_2-x_1} = \dfrac{y-y_1}{y_2-y_1} = \dfrac{z-z_1}{z_2-z_1}$ 為直線方程式



[Exercises] 1. 設$\mathbf{R}(t) = 2t\mathbf{i} - \cos 3t\mathbf{j} + t^3\mathbf{k}$，求$\dfrac{d}{dt}\mathbf{R}(t)$的單位向量. [98宜蘭電子10]

[Answers] 1. $\dfrac{2\mathbf{i}+3\sin 3t\mathbf{j}+3t^2\mathbf{k}}{\sqrt{4+9\sin^2 3t+9t^4}}$

II. 內積(點積或純量積)

定義： $\mathbf{a}\cdot\mathbf{b}=ab\cos\theta \Rightarrow \mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{a}$

其中θ為 **a** 與 **b** 始點相接時所夾的角度，且$0\le\theta\le\pi$。由定義知

性質：$\mathbf{a}\cdot\mathbf{a}=a^2$，即向量本身內積為本身長度的平方

定理：兩非零向量內積為零，此兩向量必垂直；兩非零向量垂直，其內積必為零。

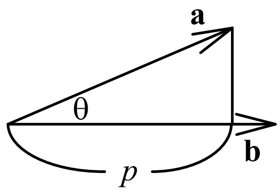
$$\mathbf{i}\cdot\mathbf{i}=1\cdot 1\cdot\cos 0^\circ=1 \Rightarrow \mathbf{j}\cdot\mathbf{j}=1，\mathbf{k}\cdot\mathbf{k}=1$$

$$\mathbf{i}\cdot\mathbf{j}=1\cdot 1\cdot\cos 90^\circ=0 \Rightarrow \mathbf{j}\cdot\mathbf{i}=\mathbf{j}\cdot\mathbf{k}=\mathbf{k}\cdot\mathbf{j}=\mathbf{k}\cdot\mathbf{i}=\mathbf{i}\cdot\mathbf{k}=0$$

以分量表示：若 $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$，$\mathbf{b}=b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$，則

$$\mathbf{a}\cdot\mathbf{b}=a_1b_1+a_2b_2+a_3b_3$$

a 在 **b** 向量方向上的投影大小為



$$p=|a\cos\theta|=|a\frac{\mathbf{a}\cdot\mathbf{b}}{ab}|=|\frac{\mathbf{a}\cdot\mathbf{b}}{b}|=|\mathbf{a}\cdot\frac{\mathbf{b}}{b}|=|\mathbf{a}\cdot\mathbf{e}_b|$$

其中$\mathbf{e}_b$是向量**b**方向的單位向量。

而**a**在**b**向量方向上的投影為$(\mathbf{a}\cdot\mathbf{e}_b)\mathbf{e}_b$。

Ex. 5

Let vector $\mathbf{F}=-\mathbf{i}+3\mathbf{j}+\mathbf{k}$, $\mathbf{G}=2\mathbf{j}-4\mathbf{k}$. Find the angle between the vectors **F** and **G**. [102勤益電子6]

[解]$\mathbf{F}\cdot\mathbf{G}=|\mathbf{F}||\mathbf{G}|\cos\theta$

$$\cos\theta=\frac{\mathbf{F}\cdot\mathbf{G}}{|\mathbf{F}||\mathbf{G}|}=\frac{(-\mathbf{i}+3\mathbf{j}+\mathbf{k})\cdot(2\mathbf{j}-4\mathbf{k})}{\sqrt{(-1)^2+3^2+1^2}\sqrt{2^2+(-4)^2}}=\frac{0+6-4}{\sqrt{11}\cdot 2\sqrt{5}}=\frac{1}{\sqrt{55}}$$

$$\theta=\cos^{-1}(\frac{1}{\sqrt{55}})$$

Ex. 6

An orthonormal basis for 3-space is {**a**, **b**, **c**}, $a=b=c=1$, any vector **v** in the space can be written as $\mathbf{v}=(\mathbf{v}\cdot\mathbf{a})\mathbf{a}+(\mathbf{v}\cdot\mathbf{b})\mathbf{b}+(\mathbf{v}\cdot\mathbf{c})\mathbf{c}$.

[解]設$\mathbf{v}=v_a\mathbf{a}+v_b\mathbf{b}+v_c\mathbf{c}\cdots\cdots$ ①

因為**a**, **b**, **c**為互相垂直的單位向量，將①式分別對**a**, **b**, **c**內積，得

$\mathbf{v}\cdot\mathbf{a}=v_a$，$\mathbf{v}\cdot\mathbf{b}=v_b$，$\mathbf{v}\cdot\mathbf{c}=v_c$，即①式可寫成

$\mathbf{v}=(\mathbf{v}\cdot\mathbf{a})\mathbf{a}+(\mathbf{v}\cdot\mathbf{b})\mathbf{b}+(\mathbf{v}\cdot\mathbf{c})\mathbf{c}$

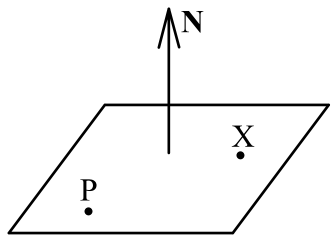
Ex. 7

Find the projection of the vector $\mathbf{v}=-2\mathbf{j}+2\mathbf{k}$ onto $\mathbf{u}=\mathbf{i}+\mathbf{j}+4\mathbf{k}$. [99中山機電III 1 (a)]

[解]$(\mathbf{v}\cdot\frac{\mathbf{u}}{|\mathbf{u}|})\frac{\mathbf{u}}{|\mathbf{u}|}=[(-2\mathbf{j}+2\mathbf{k})\cdot\frac{\mathbf{i}+\mathbf{j}+4\mathbf{k}}{\sqrt{1^2+1^2+4^2}}]\frac{\mathbf{i}+\mathbf{j}+4\mathbf{k}}{\sqrt{1^2+1^2+4^2}}=\frac{(-2+8)(\mathbf{i}+\mathbf{j}+4\mathbf{k})}{18}=\frac{\mathbf{i}+\mathbf{j}+4\mathbf{k}}{3}$

Ex. 8

Find a plane which passes through the point P(x_0, y_0, z_0) and is normal to the vector $\mathbf{N}=a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$.



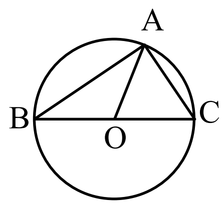
[解]設平面上任一點為X(x, y, z)，則$\overrightarrow{\mathrm{PX}}\perp\mathbf{N}\Rightarrow\mathbf{N}\cdot\overrightarrow{\mathrm{PX}}=0$

$(a\mathbf{i}+b\mathbf{j}+c\mathbf{k})\cdot[(x-x_0)\mathbf{i}+(y-y_0)\mathbf{j}+(z-z_0)\mathbf{k}]=0$

$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$為平面的方程式

Ex. 9

Prove that the triangle scribed in a semicircle is a right triangle.



[解]令$\overrightarrow{\mathrm{OC}}=\mathbf{c},\overrightarrow{\mathrm{OA}}=\mathbf{a}$，則

$\overrightarrow{\mathrm{BA}}=\overrightarrow{\mathrm{BO}}+\overrightarrow{\mathrm{OA}}=\overrightarrow{\mathrm{OC}}+\overrightarrow{\mathrm{OA}}=\mathbf{c}+\mathbf{a}$

$\overrightarrow{\mathrm{AC}}=\overrightarrow{\mathrm{AO}}+\overrightarrow{\mathrm{OC}}=-\mathbf{a}+\mathbf{c}$

$\overrightarrow{\mathrm{BA}}\cdot\overrightarrow{\mathrm{AC}}=(\mathbf{c}+\mathbf{a})\cdot(-\mathbf{a}+\mathbf{c})=c^2-a^2=0\Rightarrow\overline{\mathrm{AB}}$與$\overline{\mathrm{AC}}$互相垂直

[Exercises] 1. 有兩向量$\mathbf{A}=2\mathbf{i}+2\mathbf{j}+3\mathbf{k}$，$\mathbf{B}=4\mathbf{i}+4\mathbf{j}+6\mathbf{k}$，請計算(1)$\mathbf{A}\cdot\mathbf{B}$，(2)$\mathbf{A}\times\mathbf{B}$. [104高應大光電與通訊1]

2. If $\mathbf{A}=3\mathbf{i}-\mathbf{j}-2\mathbf{k}$，$\mathbf{B}=\mathbf{i}+2\mathbf{j}-3\mathbf{k}$，Find(1)$\mathbf{A}\cdot\mathbf{B}$，(2)the angle between **A** and **B**，(3)the projection of **A** on **B**. [103雲科大電子4(a)(c)(d)]

3. Find the equation of the plane containing the point (2, –3, 4) and orthogonal to the Vector $\mathbf{A}=8\mathbf{i}-6\mathbf{j}+4\mathbf{k}$. [103雲科大電機4(2)]

4. 試求平面$x+y+z=-1$與$x+y=2$間夾角的餘弦。[105高第一環安甲5(a)]

5. Find a unit vector perpendicular to the plane $4x+2y+4z=-7$. [105嘉大土木3]

[Answers] 1. (1)34 (2)0　　2. (1)–5 (2)$\cos^{-1}(-5/14)$ (3)$-\frac{5}{14}(\mathbf{i}+2\mathbf{j}-3\mathbf{k})$

3. $4x-3y+2z=25$　　4. $\sqrt{6}/3$　　5. $\pm(2\mathbf{i}+\mathbf{j}+2\mathbf{k})/3$

III. 外積(叉積或向量積)

定義：向量 **a** 與 **b** 的外積為

$$\mathbf{v}=\mathbf{a}\times\mathbf{b}$$

v 的大小為$v=ab\sin\theta$，其中θ為 **a** 與 **b** 始點相接時的夾角，且$0\le\theta\le\pi$；**v** 的方向為將右手掌由 **a** 轉至 **b** 時拇指所指的方向，且$\mathbf{a}\times\mathbf{b}$同時與 **a** 及 **b** 垂直。

$$\because \mathbf{i}\times\mathbf{i}=\mathbf{j}\times\mathbf{j}=\mathbf{k}\times\mathbf{k}=0,$$

$$\mathbf{i}\times\mathbf{j}=\mathbf{k},\quad \mathbf{j}\times\mathbf{k}=\mathbf{i},\quad \mathbf{k}\times\mathbf{i}=\mathbf{j},$$

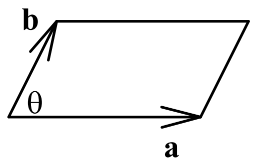
$$\mathbf{j}\times\mathbf{i}=-\mathbf{k},\quad \mathbf{k}\times\mathbf{j}=-\mathbf{i},\quad \mathbf{i}\times\mathbf{k}=-\mathbf{j}$$

以分量表示：若$\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k},\ \mathbf{b}=b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k},\ \mathbf{v}=v_1\mathbf{i}+v_2\mathbf{j}+v_3\mathbf{k}$

$$\therefore \mathbf{a}\times\mathbf{b}=(a_2b_3-a_3b_2)\mathbf{i}+(a_3b_1-a_1b_3)\mathbf{j}+(a_1b_2-a_2b_1)\mathbf{k}$$

或

$$\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$



性質：1. $\mathbf{a}\times\mathbf{b}=-\mathbf{b}\times\mathbf{a}$

2. $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}\neq\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$

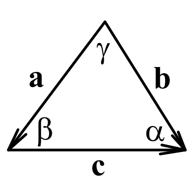
3. 若$\mathbf{a}\times\mathbf{b}=0$，則(1)$a=0$或$b=0$。

(2)**a** 與 **b** 平行。

4. $|\mathbf{a}\times\mathbf{b}|=ab\sin\theta$ 為圖示 **a** 與 **b** 為鄰邊所成平行四邊形的面積

Ex. 10

Derive the law of sines using vectors.



[解]$\mathbf{c}=\mathbf{b}-\mathbf{a}\Rightarrow\mathbf{c}\times\mathbf{c}=\mathbf{c}\times(\mathbf{b}-\mathbf{a})\Rightarrow 0=\mathbf{c}\times\mathbf{b}-\mathbf{c}\times\mathbf{a}\Rightarrow\mathbf{c}\times\mathbf{b}=\mathbf{c}\times\mathbf{a}$

$cb\sin\alpha=ca\sin(\pi-\beta)\Rightarrow\dfrac{a}{\sin\alpha}=\dfrac{b}{\sin\beta}$，同理$\dfrac{a}{\sin\alpha}=\dfrac{b}{\sin\beta}=\dfrac{c}{\sin\gamma}$

Ex. 11

已知空間中三點A(1, 1, 1), B(2, 3, 4), C(3, 0, −1)，試求(1)向量 $\overrightarrow{AB}$ 與 $\overrightarrow{BC}$ 之外積(Cross Product)；(2)ABC構成之三角形面積。[103雲科大環安8]

[解] $\overrightarrow{AB}=\mathbf{i}+2\mathbf{j}+3\mathbf{k}$，$\overrightarrow{BC}=\mathbf{i}-3\mathbf{j}-5\mathbf{k}$

$$(1)\overrightarrow{AB}\times\overrightarrow{BC}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&2&3\\1&-3&-5\end{vmatrix}=-\mathbf{i}+8\mathbf{j}-5\mathbf{k}$$

$$(2)\Delta ABC=\frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{BC}|=\frac{1}{2}\sqrt{(-1)^2+8^2+(-5)^2}=\frac{3\sqrt{10}}{2}$$

Ex. 12

空間有三點A(2, 5, 7), B(1, 3, 4), C(4, 5, 5)，(1)以向量方式求解包含此三點之平面方程式。(2)求此三點圍成之三角形面積。[104高應大機械甲丙4]

[解] $\overrightarrow{AB}=-\mathbf{i}-2\mathbf{j}-3\mathbf{k}$, $\overrightarrow{AC}=2\mathbf{i}-2\mathbf{k}$

$$(1)\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-1&-2&-3\\2&0&-2\end{vmatrix}=4\mathbf{i}-8\mathbf{j}+4\mathbf{k}\Rightarrow \text{平面的法向量為}\ \mathbf{n}=\mathbf{i}-2\mathbf{j}+\mathbf{k}$$

設平面上任一點為 $X(x, y, z)$，則

$\mathbf{n}\perp\overrightarrow{AX}=0\Rightarrow\mathbf{n}\cdot\overrightarrow{AX}=0$

$(\mathbf{i}-2\mathbf{j}+\mathbf{k})\cdot[(x-2)\mathbf{i}+(y-5)\mathbf{j}+(z-7)\mathbf{k}]=0$

平面方程式為$(x-2)-2(y-5)+(z-7)=0\Rightarrow x-2y+z+1=0$

(2)包含此三點的三角形面積為

$$\frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|=\frac{1}{2}\sqrt{4^2+(-8)^2+4^2}=2\sqrt{6}$$

Ex. 13

Find the distance of two lines L_1: $\frac{x-2}{1}=\frac{y-5}{6}=\frac{z-5}{2}$, $L_2: \frac{x+4}{-1}=\frac{y+2}{3}=\frac{z+4}{4}$.

[解](1)L_1的方向向量為$\mathbf{v}_1=\mathbf{i}+6\mathbf{j}+2\mathbf{k}$，$L_2$的方向向量為$\mathbf{v}_2=-\mathbf{i}+3\mathbf{j}+4\mathbf{k}$

L_1上任一點 $P(m+2, 6m+5, 2m+5)$，L_2上的點任一點 $Q(-n-4, 3n-2, 4n-4)$

$\overrightarrow{PQ}\cdot\mathbf{v}_1=0\Rightarrow[(-n-m-6)\mathbf{i}+(3n-6m-7)\mathbf{j}+(4n-2m-9)\mathbf{k}]\cdot(\mathbf{i}+6\mathbf{j}+2\mathbf{k})=0$

$(-n-m-6)+6(3n-6m-7)+2(4n-2m-9)=0\Rightarrow 25n-41m=66\cdots\cdots$①

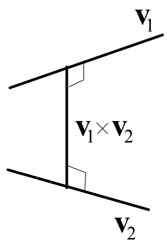
$\overrightarrow{PQ}\cdot\mathbf{v}_2=0\Rightarrow[(-n-m-6)\mathbf{i}+(3n-6m-7)\mathbf{j}+(4n-2m-9)\mathbf{k}]\cdot(-\mathbf{i}+3\mathbf{j}+4\mathbf{k})=0$

$-(-n-m-6)+3(3n-6m-7)+4(4n-2m-9)=0\Rightarrow 26n-25m=51\cdots\cdots$②

①$\times 26-$②$\times 25$： $-441m=441\Rightarrow m=-1$，代入①得 $n=1$

m, n 代回得 $P(1, -1, 3)$, $Q(-5, 1, 0)$

二歪斜線的距離為 $\overline{PQ}=\sqrt{(-6)^2+2^2+(-3)^2}=7$



(2)L_1的方向向量為 $\mathbf{v}_1=\mathbf{i}+6\mathbf{j}+2\mathbf{k}$，$L_2$的方向向量為 $\mathbf{v}_2=-\mathbf{i}+3\mathbf{j}+4\mathbf{k}$

$A(2, 5, 5)$, $B(-4, -2, -4)$分別在直線 L_1, L_2上

同時與 $\mathbf{v}_1$, $\mathbf{v}_2$垂直的向量為

$$\mathbf{v}_1\times\mathbf{v}_2=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&6&2\\-1&3&4\end{vmatrix}=18\mathbf{i}-6\mathbf{j}+9\mathbf{k}=3(6\mathbf{i}-2\mathbf{j}+3\mathbf{k})$$

L_1, L_2 的距離為$\left|\overrightarrow{AB}\cdot\frac{\mathbf{v}_1\times\mathbf{v}_2}{|\mathbf{v}_1\times\mathbf{v}_2|}\right|=\left|(-6\mathbf{i}-7\mathbf{j}-9\mathbf{k})\cdot\frac{3(6\mathbf{i}-2\mathbf{j}+3\mathbf{k})}{3\sqrt{6^2+(-2)^2+3^2}}\right|$

$$=\frac{|-36+14-27|}{7}=7$$

Ex. 14

In the three-dimensional Cartesian coordinates, find the shortest distance of the intersection of $x+2y+3z=6$ and $3x+2y+z=6$ to the origin (0, 0, 0). Also, what are the coordinates of the point that has this shortest distance? [98 交大機械丁 5]

[解]設兩平面交線的方向向量為$\mathbf{v}=a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$，則$\mathbf{v}$同時垂直兩平面的法向量

$$\begin{cases}\mathbf{v}\cdot(\mathbf{i}+2\mathbf{j}+3\mathbf{k})=0\Rightarrow a+2b+3c=0\\ \mathbf{v}\cdot(3\mathbf{i}+2\mathbf{j}+\mathbf{k})=0\Rightarrow 3a+2b+c=0\end{cases}\Rightarrow a:b:c=\begin{vmatrix}2&3\\2&1\end{vmatrix}:\begin{vmatrix}3&1\\1&3\end{vmatrix}:\begin{vmatrix}1&2\\3&2\end{vmatrix}=1:(-2):1$$

任意找一點同時在兩平面上：$(1,1,1)$，因此直線的參數式為$x=t+1,\ y=-2t+1, z=t+1$

直線上任一點與原點的距離為

$\sqrt{(t+1)^2+(-2t+1)^2+(t+1)^2}=\sqrt{6t^2+3}$

當$t=0$時，距離最短為$\sqrt{3}$，對應的點為$(1,1,1)$

[Exercises] 1. Find the area of triangle with vertices (1, 0, 2), (3, 2, 1), (2, 1, 3).

[99中山機電III 1 (b)]

2. 求過三點$P_1(1, -1, 2)$, $P_2(3, 0, 0)$, $P_3(4, 2, 1)$之平面方程式。

[105高第一環安甲5(b)]

3. 向量$\mathbf{A}=\mathbf{i}+\mathbf{j}+\mathbf{k}$與向量$\mathbf{B}=\mathbf{j}-\mathbf{k}$所構成之平面的法向量與向量$\mathbf{C}=\mathbf{i}-2\mathbf{k}$的夾角為何？

4. Find the equation of the line containing (1, 4, 3) which is perpendicular to both of the lines $\dfrac{x-1}{2}=y+3=\dfrac{z-2}{4}$ and $\dfrac{x+2}{3}=\dfrac{y-4}{2}=\dfrac{z+1}{-2}$.

5. Find parametric equations for the intersection of the planes $2x-y+z=-2$ and $x+y+z=0$.

6. Write the equation of the plane containing the lines $x=y=\dfrac{4-z}{4}$ and $2x=2-y=z$.

[Answers]1. $3\sqrt{2}/2$ 2. $5x-4y+3z=15$ 3. $\cos^{-1}\dfrac{4}{\sqrt{30}}$ 或$-\cos^{-1}\dfrac{4}{\sqrt{30}}$

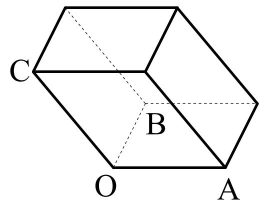
4. $\dfrac{x-1}{-10}=\dfrac{y-4}{16}=z-3$ 5. $x=2t$, $y=1+t$, $z=-1-3t$ 6. $2x+2y+z=4$

IV. 多重積

1. 純量三重積$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$

考慮一平行六面體，分別以$\overrightarrow{OA}=\mathbf{a}$、$\overrightarrow{OB}=\mathbf{b}$、$\overrightarrow{OC}=\mathbf{c}$為鄰邊，如圖所示。因為$\mathbf{a}\times\mathbf{b}$的值為$\mathbf{a}$與$\mathbf{b}$為鄰邊的平行四邊形面積，而$\mathbf{a}\times\mathbf{b}$同時與$\mathbf{a}$及$\mathbf{b}$垂直，即垂直圖中上下平行四邊形，又$\mathbf{c}$在$\mathbf{a}\times\mathbf{b}$方向的投影為$\mathbf{c}\cdot\dfrac{\mathbf{a}\times\mathbf{b}}{|\mathbf{a}\times\mathbf{b}|}$，為圖中上下平行四邊形的距離，即$\mathbf{a}$、$\mathbf{b}$及$\mathbf{c}$為鄰邊，上下平行四邊形當底時，該平行六面體的高，因此，此平行六面體的體積為

$$底\times高=|\mathbf{a}\times\mathbf{b}||\mathbf{c}\cdot\frac{\mathbf{a}\times\mathbf{b}}{|\mathbf{a}\times\mathbf{b}|}|=|\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})|$$



由內積可交換及以分量表示純量三重積知

$$\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})=(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}\cdot(c_1\mathbf{i}+c_2\mathbf{j}+c_3\mathbf{k})=\begin{vmatrix}c_1 & c_2 & c_3\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$

由行列式運算規則知

$$\begin{vmatrix}c_1 & c_2 & c_3\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}=\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}=(\mathbf{b}\times\mathbf{c})\cdot\mathbf{a}=\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$$

同理得

$$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}=\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})=(\mathbf{b}\times\mathbf{c})\cdot\mathbf{a}=\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=(\mathbf{c}\times\mathbf{a})\cdot\mathbf{b}=\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})$$

因此，在純量三重積中，點跟叉的位置可以互換，經常以$(\mathbf{a}\ \mathbf{b}\ \mathbf{c})$來表示純量三重積為

$$(\mathbf{a}\ \mathbf{b}\ \mathbf{c})=\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}$$

Ex. 15

One corner of a rectangular parallelepiped is at (−1, 2, 2) and three incident sides extend from this point to (0, 1, 1), (–4, 6, 8), and (–3, –2, 4). Please find the volume of this parallelepiped .[98 彰師大車輛 3]

[解]設 A(−1, 2, 2), B(0, 1, 1), C(−4, 6, 8), D(−3, − 2, 4)

$\overrightarrow{AB} = \mathbf{i} - \mathbf{j} - \mathbf{k},\ \overrightarrow{AC} = -3\mathbf{i} + 4\mathbf{j} + 6\mathbf{k},\ \overrightarrow{AD} = -2\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$

平行六面體的體積為

$$\left\| \begin{matrix} 1 & -1 & -1 \\ -3 & 4 & 6 \\ -2 & -4 & 2 \end{matrix} \right\| = |8 - 12 + 12 - 8 + 24 - 6| = 18$$

2. 向量三重積$(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$

因為$\mathbf{a}\times\mathbf{b}$與$\mathbf{a}$、$\mathbf{b}$所在的平面垂直，而$(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$垂直$\mathbf{a}\times\mathbf{b}$，因此$(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$落在$\mathbf{a}$與$\mathbf{b}$的平面，可設

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = m\mathbf{a} + n\mathbf{b}$$

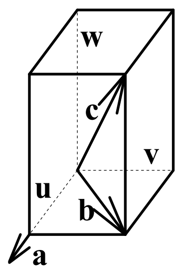
又$(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$與$\mathbf{c}$垂直。所以

$$[(\mathbf{a}\times\mathbf{b})\times\mathbf{c}]\cdot\mathbf{c} = m\mathbf{a}\cdot\mathbf{c} + n\mathbf{b}\cdot\mathbf{c} \Rightarrow 0 = m\mathbf{a}\cdot\mathbf{c} + n\mathbf{b}\cdot\mathbf{c}$$

令$n = \lambda\mathbf{a}\cdot\mathbf{c}$， $m = -\lambda\mathbf{b}\cdot\mathbf{c}$，則

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = \lambda[(\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{b}\cdot\mathbf{c})\mathbf{a}] \tag{1.1}$$

為求得λ，設$\mathbf{u}$為與$\mathbf{a}$平行的單位向量，$\mathbf{v}$與$\mathbf{a}$垂直且落在$\mathbf{a}$與$\mathbf{b}$的平面上的單位向量，而$\mathbf{w} = \mathbf{u}\times\mathbf{v}$，則



$$\mathbf{a} = a_1\mathbf{u},\quad \mathbf{b} = b_1\mathbf{u} + b_2\mathbf{v},\quad \mathbf{c} = c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{w}$$

代入(1.1)式

$$[a_1\mathbf{u}\times(b_1\mathbf{u} + b_2\mathbf{v})]\times(c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{w}) = \lambda\{[a_1\mathbf{u}\cdot(c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{w})](b_1\mathbf{u} + b_2\mathbf{v}) - [(b_1\mathbf{u} + b_2\mathbf{v})\cdot(c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{w})](a_1\mathbf{u})\}$$

$a_1b_2\mathbf{w}\times(c_1\mathbf{u}+c_2\mathbf{v}+c_3\mathbf{w})=\lambda[a_1c_1(b_1\mathbf{u}+b_2\mathbf{v})-(b_1c_1+b_2c_2)(a_1\mathbf{u})]$

$a_1b_2c_1\mathbf{v}-a_1b_2c_2\mathbf{u}=\lambda(a_1b_2c_1\mathbf{v}-a_1b_2c_2\mathbf{u})$

得 $\lambda=1$

因此

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c}=(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{b}\cdot\mathbf{c})\mathbf{a}$$

同理

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}$$

[註] $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}\neq\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$

[Exercises] 1. Vector $\mathbf{A}=8\mathbf{i}-6\mathbf{j}+4\mathbf{k}$, $\mathbf{B}=\mathbf{i}+\mathbf{j}$, $\mathbf{C}=\mathbf{i}-\mathbf{j}$, find $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$. [103雲科大電機4(1)]

2. 求以$\mathbf{a}=2\mathbf{i}-\mathbf{j}-\mathbf{k}$，$\mathbf{b}=\mathbf{j}+7\mathbf{k}$，$\mathbf{c}=\mathbf{i}+2\mathbf{k}$為三鄰邊所形成平行六面體的體積。[104勤益機械7]

[Answers] 1. –8　2. 2

第二章 向量的微分

I. 向量的導數

定義：若以下的極限存在

$$\mathbf{v}'(t)=\lim_{\Delta t\to 0}\frac{\mathbf{v}(t+\Delta t)-\mathbf{v}(t)}{\Delta t}$$

則向量函數$\mathbf{v}(t)$可被微分，向量$\mathbf{v}'(t)$稱為$\mathbf{v}(t)$的導數，以卡式坐標系統的分量表示，導數$\mathbf{v}'(t)$可由每個分量的微分表示為

$$\mathbf{v}'(t)=v'_1(t)\mathbf{i}+v'_2(t)\mathbf{j}+v'_3(t)\mathbf{k}$$

關係式

$$(\mathbf{u}+\mathbf{v})'=\mathbf{u}'+\mathbf{v}'$$

$$(\mathbf{u}\cdot\mathbf{v})'=\mathbf{u}'\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{v}'$$

$$(\mathbf{u}\times\mathbf{v})'=\mathbf{u}'\times\mathbf{v}+\mathbf{u}\times\mathbf{v}'$$

Ex. 1

Let $\mathbf{v}(t)$ be a vector function, whose length is constant. Then $\mathbf{v}\cdot\mathbf{v}=v^2 \Rightarrow (\mathbf{v}\cdot\mathbf{v})'=0$, we get $2\mathbf{v}\cdot\mathbf{v}'=0$. This yields the important result: the derivative of a vector function $\mathbf{v}(t)$ of constant length is either the zero vector or is perpendicular to $\mathbf{v}(t)$.

II. 空間曲線的幾何

1. 弧長

在卡式坐標系統中，一曲線C可以表成一向量函數，稱為位置向量

$$\mathbf{r}(t)=x(t)\,\mathbf{i}+y(t)\,\mathbf{j}+z(t)\,\mathbf{k}$$

$$\mathbf{v}(t)=\mathbf{r}'(t)=x'(t)\,\mathbf{i}+y'(t)\,\mathbf{j}+z'(t)\,\mathbf{k}$$

若s表示沿著曲線的弧長，則

$$\mathbf{v}(t)=\mathbf{r}'(t)=\frac{d\mathbf{r}}{ds}\frac{ds}{dt}=(\frac{dx}{ds}\mathbf{i}+\frac{dy}{ds}\mathbf{j}+\frac{dz}{ds}\mathbf{k})\frac{ds}{dt}$$

括號表示曲線某一點的切線單位向量，以**T**表示為

$$\mathbf{T}=\frac{dx}{ds}\mathbf{i}+\frac{dy}{ds}\mathbf{j}+\frac{dz}{ds}\mathbf{k}=\frac{d\mathbf{r}}{ds}=\frac{\mathbf{v}}{v}$$

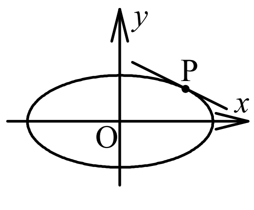
$$|\mathbf{v}|=|\mathbf{T}|\frac{ds}{dt}=\frac{ds}{dt}$$

弧長為

$$s=\int ds=\int|\mathbf{v}|dt=\int\sqrt{\mathbf{v}\cdot\mathbf{v}}\,dt \qquad (2.1)$$

Ex. 2

Find the tangent to the ellipse $\frac{1}{4}x^2+y^2=1$ at P$(\sqrt{2},\frac{1}{\sqrt{2}})$. [100彰師大積體電路7]



[解]$\mathbf{r}(t)=2\cos t\,\mathbf{i}+\sin t\,\mathbf{j}$，由$2\cos t=\sqrt{2},\ \sin t=\frac{1}{\sqrt{2}}\Rightarrow$ P點對應$t=\frac{\pi}{4}$

$\mathbf{r}'(t)=-2\sin t\,\mathbf{i}+\cos t\,\mathbf{j}\Rightarrow\mathbf{r}'(\pi/4)=-\sqrt{2}\mathbf{i}+\frac{1}{\sqrt{2}}\mathbf{j}\Rightarrow$ 斜率為$\frac{\frac{1}{\sqrt{2}}}{-\sqrt{2}}=-\frac{1}{2}$

切線為$y-\frac{1}{\sqrt{2}}=-\frac{1}{2}(x-\sqrt{2})\Rightarrow x+2y=2\sqrt{2}$

Ex. 3

A curve is defined as $\mathbf{r}(t)=[a\cos t,\ a\sin t,\ ct]$, Please find $\mathbf{r}(s)$, where s is the arc length. [96暨南土木4(a)]

[解]$\mathbf{v}(t)=-a\sin t\,\mathbf{i}+a\cos t\,\mathbf{j}+c\,\mathbf{k}$

$$\mathbf{v}\cdot\mathbf{v}=a^2+c^2$$

$$s=\int_0^t\sqrt{a^2+c^2}\,dt=t\sqrt{a^2+c^2}\Rightarrow t=\frac{s}{\sqrt{a^2+c^2}}$$

$$\mathbf{r}(s)=a\cos\frac{s}{\sqrt{a^2+c^2}}\mathbf{i}+a\sin\frac{s}{\sqrt{a^2+c^2}}\mathbf{j}+c\frac{s}{\sqrt{a^2+c^2}}\mathbf{k}$$

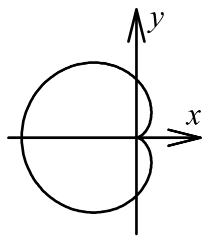
Ex. 4

Find the length of the space curve: $y=\sin 2\pi x$, $z=\cos 2\pi x$, from (0, 0, 1) to (1, 0, 1).

[解]$\mathbf{r}=x\,\mathbf{i}+\sin 2\pi x\,\mathbf{j}+\cos 2\pi x\,\mathbf{k}\Rightarrow d\mathbf{r}/dx=\mathbf{i}+2\pi\cos 2\pi x\,\mathbf{j}-2\pi\sin 2\pi x\,\mathbf{k}$

$$s=\int_0^1\sqrt{1^2+(2\pi\cos 2\pi x)^2+(2\pi\sin 2\pi x)^2}\,dx=\int_0^1\sqrt{1+4\pi^2}\,dx=\sqrt{1+4\pi^2}$$

Ex. 5



Find the length of the curve: $r=a(1-\cos\theta),\ 0\le\theta\le 2\pi,\ a>0$.

[解]$\mathbf{r}=r\mathbf{e}_r\Rightarrow\dfrac{d\mathbf{r}}{d\theta}=\dfrac{dr}{d\theta}\mathbf{e}_r+r\dfrac{d\mathbf{e}_r}{d\theta}=a\sin\theta\,\mathbf{e}_r+r\,\mathbf{e}_\theta=a\sin\theta\,\mathbf{e}_r+a(1-\cos\theta)\,\mathbf{e}_\theta$

$$s=\int_0^{2\pi}\sqrt{(a\sin\theta)^2+[a(1-\cos\theta)]^2}\,d\theta=\int_0^{2\pi}\sqrt{2a^2-2a^2\cos\theta}\,d\theta=a\int_0^{2\pi}\sqrt{2(1-\cos\theta)}\,d\theta$$

$$=a\int_0^{2\pi}2\sin\frac{\theta}{2}d\theta=4a(-\cos\frac{\theta}{2})\Big|_0^{2\pi}=8a$$

2. Frenet公式

已知切線單位向量$\mathbf{T}=\dfrac{d\mathbf{r}}{ds}$，定義

$$\frac{d\mathbf{T}}{ds}=\kappa\mathbf{N},\quad \kappa\geq 0 \tag{2.2}$$

其中κ是$\dfrac{d\mathbf{T}}{ds}$的大小稱為曲率，**N**與**T**垂直，稱為該曲線的法線單位向量。曲率的倒數$\rho=\dfrac{1}{\kappa}$稱為曲率半徑。再定義第三個單位向量同時與**T**及**N**垂直，稱為副法線單位向量為

$$\mathbf{B}=\mathbf{T}\times\mathbf{N}$$

而

$$\frac{d\mathbf{B}}{ds}=\frac{d}{ds}(\mathbf{T}\times\mathbf{N})=\frac{d\mathbf{T}}{ds}\times\mathbf{N}+\mathbf{T}\times\frac{d\mathbf{N}}{ds}=\kappa\mathbf{N}\times\mathbf{N}+\mathbf{T}\times\frac{d\mathbf{N}}{ds}=\mathbf{T}\times\frac{d\mathbf{N}}{ds}$$

因為$\dfrac{d\mathbf{B}}{ds}$垂直**B**且由右手螺旋知$\dfrac{d\mathbf{B}}{ds}$亦與**T**垂直，$\dfrac{d\mathbf{B}}{ds}$可寫成

$$\frac{d\mathbf{B}}{ds}=-\tau\mathbf{N} \tag{2.3}$$

其中τ稱為曲線的扭率，又

$$\frac{d\mathbf{N}}{ds}=\frac{d}{ds}(\mathbf{B}\times\mathbf{T})=\frac{d\mathbf{B}}{ds}\times\mathbf{T}+\mathbf{B}\times\frac{d\mathbf{T}}{ds}=-\tau\mathbf{N}\times\mathbf{T}+\mathbf{B}\times\kappa\mathbf{N}=-\kappa\mathbf{T}+\tau\mathbf{B} \tag{2.4}$$

(2.2)、(2.3)及(2.4)式稱為Frenet公式。
通常位置向量**r**為*t*的函數，為求得κ及τ，將**r**對*t*微分

$$\dot{\mathbf{r}}=\frac{d\mathbf{r}}{dt}=\frac{d\mathbf{r}}{ds}\frac{ds}{dt}=v\mathbf{T}$$

$$\ddot{\mathbf{r}}=\dot{v}\mathbf{T}+v\frac{d\mathbf{T}}{dt}=\dot{v}\mathbf{T}+v\frac{d\mathbf{T}}{ds}\frac{ds}{dt}=\dot{v}\mathbf{T}+v(\kappa\mathbf{N})v=\dot{v}\mathbf{T}+\kappa v^2\mathbf{N}$$

$$\begin{aligned}\dddot{\mathbf{r}}&=\ddot{v}\mathbf{T}+\dot{v}\frac{d\mathbf{T}}{dt}+\frac{d}{dt}(\kappa v^2)\mathbf{N}+\kappa v^2\frac{d\mathbf{N}}{dt}=\ddot{v}\mathbf{T}+\dot{v}\kappa v\mathbf{N}+\frac{d}{dt}(\kappa v^2)\mathbf{N}+\kappa v^3(-\kappa\mathbf{T}+\tau\mathbf{B})\\&=(\ddot{v}-\kappa^2v^3)\mathbf{T}+[\kappa v\dot{v}+\frac{d}{dt}(\kappa v^2)]\mathbf{N}+\kappa v^3\tau\mathbf{B}\end{aligned}$$

$$\dot{\mathbf{r}}\times\ddot{\mathbf{r}}=\kappa v^3\mathbf{B}\Rightarrow\kappa=\frac{|\dot{\mathbf{r}}\times\ddot{\mathbf{r}}|}{|\dot{\mathbf{r}}|^3}\qquad(\dot{\mathbf{r}}\times\ddot{\mathbf{r}})\cdot\dddot{\mathbf{r}}=\kappa^2v^6\tau\Rightarrow\tau=\frac{(\dot{\mathbf{r}}\ \ddot{\mathbf{r}}\ \dddot{\mathbf{r}})}{|\dot{\mathbf{r}}\times\ddot{\mathbf{r}}|^2} \tag{2.5}$$

Ex. 6

A curve is defined as $\mathbf{r}(t)=[a\cos t,\ a\sin t,\ ct]$, Please find $\mathbf{u}(s)$, where s is the arc length and $\mathbf{u}(s)$ is the unit tangent vector; $\kappa(s)$, where $\kappa(s)$ is the curvature of the curve; $\mathbf{p}(s)$, where $\mathbf{p}(s)$ is the unit principle normal vector; $\mathbf{b}(s)$, where $\mathbf{b}(s)$ is the unit binormal vector; $\tau(s)$, where $\tau(s)$ is the torsion of the curve. [96暨南土木4(b)(c)(d)(e)(f)]

[解] $\mathbf{v}=\dot{\mathbf{r}}=-a\sin t\mathbf{i}+a\cos t\mathbf{j}+c\mathbf{k}=v\mathbf{u}\Rightarrow v=\sqrt{a^2+c^2},\ \dot{v}=0$

$$s=\int_0^t v\,dt=\int_0^t\sqrt{a^2+c^2}\,dt=t\sqrt{a^2+c^2}\Rightarrow t=\frac{s}{\sqrt{a^2+c^2}}$$

$$\mathbf{u}=\frac{\mathbf{v}}{v}=\frac{-a\sin t}{\sqrt{a^2+c^2}}\mathbf{i}+\frac{a\cos t}{\sqrt{a^2+c^2}}\mathbf{j}+\frac{c}{\sqrt{a^2+c^2}}\mathbf{k}$$

$$=\frac{1}{\sqrt{a^2+c^2}}\left(-a\sin\frac{s}{\sqrt{a^2+c^2}}\mathbf{i}+a\cos\frac{s}{\sqrt{a^2+c^2}}\mathbf{j}+c\mathbf{k}\right)$$

$$\mathbf{a}=\ddot{\mathbf{r}}=-a\cos t\mathbf{i}-a\sin t\mathbf{j}=\dot{v}\mathbf{u}+\kappa v^2\mathbf{p}=\kappa(a^2+c^2)\mathbf{p}\Rightarrow\dddot{\mathbf{r}}=a\sin t\mathbf{i}-a\cos t\mathbf{j}$$

$$\dot{\mathbf{r}}\times\ddot{\mathbf{r}}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-a\sin t&a\cos t&c\\-a\cos t&-a\sin t&0\end{vmatrix}=\kappa v^3\mathbf{b}\Rightarrow ac\sin t\mathbf{i}-ac\cos t\mathbf{j}+a^2\mathbf{k}=\kappa v^3\mathbf{b}$$

$$\kappa v^3=\sqrt{(ac\sin t)^2+(-ac\cos t)^2+(a^2)^2}=\sqrt{a^2c^2+a^4}=a\sqrt{a^2+c^2}\Rightarrow\kappa=\frac{a}{a^2+c^2}$$

代入$\mathbf{a}$：$-a\cos t\mathbf{i}-a\sin t\mathbf{j}=\dfrac{a}{a^2+c^2}\cdot(a^2+c^2)\mathbf{p}$

$$\mathbf{p}=-\cos t\mathbf{i}-\sin t\mathbf{j}=-\cos\frac{s}{\sqrt{a^2+c^2}}\mathbf{i}-\sin\frac{s}{\sqrt{a^2+c^2}}\mathbf{j}$$

$$\tau=\frac{(\dot{\mathbf{r}}\ \ddot{\mathbf{r}}\ \dddot{\mathbf{r}})}{|\dot{\mathbf{r}}\times\ddot{\mathbf{r}}|^2}=\frac{a^2c}{a^2(a^2+c^2)}=\frac{c}{a^2+c^2}$$

$$\mathbf{b}=\mathbf{u}\times\mathbf{p}=\frac{1}{\sqrt{a^2+c^2}}\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-a\sin t&a\cos t&c\\-\cos t&-\sin t&0\end{vmatrix}=\frac{1}{\sqrt{a^2+c^2}}(c\sin t\mathbf{i}-c\cos t\mathbf{j}+a\mathbf{k})$$

$$=\frac{1}{\sqrt{a^2+c^2}}\left(c\sin\frac{s}{\sqrt{a^2+c^2}}\mathbf{i}-c\cos\frac{s}{\sqrt{a^2+c^2}}\mathbf{j}+a\mathbf{k}\right)$$

Ex. 7

Show that for a curve $y=y(x)$ in the x-y plane, $\kappa(x)=\dfrac{|y''|}{(1+y'^2)^{3/2}}(y'=\dfrac{dy}{dx})$

[解]$\mathbf{r}=x\,\mathbf{i}+y\,\mathbf{j}$

$\mathbf{v}=\mathbf{i}+y'\,\mathbf{j}=v\,\mathbf{T},\Rightarrow v=\sqrt{1+y'^2}$

$\mathbf{a}=y''\mathbf{j}=\dot{v}\,\mathbf{T}+\kappa v^2\mathbf{N}$

$\mathbf{v}\times\mathbf{a}:(\mathbf{i}+y'\,\mathbf{j})\times y''\mathbf{j}=v\,\mathbf{T}\times(\dot{v}\,\mathbf{T}+\kappa v^2\mathbf{N})\Rightarrow y''\mathbf{k}=\kappa v^3\,\mathbf{B}$

$|y''|=\kappa v^3\Rightarrow\kappa=\dfrac{|y''|}{v^3}=\dfrac{|y''|}{(1+y'^2)^{3/2}}$

[Exercise]1.某質點的位移場為$\mathbf{u}=2\cos t\mathbf{i}+2\sin t\mathbf{j}+2.237t\mathbf{k}$(m)，時間單位秒。試求(1)$t=\pi/6$秒之加速度；(2)時間0至10秒質點運動的曲線長度。[103勤益機械3]

[Answer] 1. (1) $-\sqrt{3}\,\mathbf{i}-\mathbf{j}$ m/s^2 (2)30 m

III. 梯度(Gradient)

定義：一純量函數 $f(x, y, z)$的梯度為一個向量函數，定義為

$$\text{grad}\, f = \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

其中$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$ 為一微分運算子(differential operator)，讀成*del*，而f 沿著曲線切線方向的方向導數(directional derivative)為

$$\frac{df}{ds} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds} + \frac{\partial f}{\partial z}\frac{dz}{ds} = \nabla f \cdot \frac{d\mathbf{r}}{ds} = \nabla f \cdot \mathbf{T}$$

其中**T**曲線的切線單位向量。因為$\nabla f \cdot \mathbf{T} = |\nabla f| \cos\theta$，其中θ為$\nabla f$ 與**T**的夾角，得$\frac{df}{ds} = |\nabla f| \cos\theta$，因此$\nabla f$在**T**方向的分量就是$f$在該方向的方向導數，而且$\nabla f$ 是$\frac{df}{ds}$最大值的方向，$\frac{df}{ds}$的最大值為$|\nabla f|$。

考慮$f(x, y, z) =$常數的曲面，當我們沿著此曲面上任一條曲線C，f 一直是定值，知$\frac{df}{ds} = 0$，因此

$$\frac{df}{ds} = \nabla f \cdot \frac{d\mathbf{r}}{ds} = 0 \Rightarrow \nabla f \perp \frac{d\mathbf{r}}{ds} \Rightarrow \nabla f \perp \mathbf{T}$$

其中$\mathbf{r}(s)$是沿著曲線C的位置向量，因為∇f完全由f所決定，且C是曲面上任一條曲線，因此∇f 必垂直該曲面。

Ex. 8

Please find the unit normal vector of surface $xz^2-2xy-6x=8$ at point P(1, −1, 2), and please find the tangent plane at that point. [100嘉義土木與水資源3]

[解]令$f(x, y, z)=xz^2-2xy-6x$，則$f=8$為該曲面，

$\nabla f=(z^2-2y-6)\,\mathbf{i}-2x\,\mathbf{j}+2xz\,\mathbf{k}\Rightarrow \nabla f|_{(1,\,-1,\,2)}=-2\mathbf{j}+4\mathbf{k}$

在P點垂直該曲面的單位向量為

$$\mathbf{n}=\left.\frac{\nabla f}{|\nabla f|}\right|_{(1,-1,2)}=\frac{-2\mathbf{j}+4\mathbf{k}}{\sqrt{(-2)^2+4^2}}=-\frac{1}{\sqrt{5}}\mathbf{j}+\frac{2}{\sqrt{5}}\mathbf{k}$$

過P點的切平面為

$$(-2\mathbf{j}+4\mathbf{k})\cdot[(x-1)\mathbf{i}+(y+1)\mathbf{j}+(z-2)\mathbf{k}]=0\Rightarrow(y+1)+2(z-2)=0\Rightarrow y-2z+5=0$$

Ex. 9

若純量場$\varphi(x, y, z)=1-x^2-y^2-xyz$、向量$\mathbf{w}=\mathbf{i}+\mathbf{j}+\mathbf{k}$、點$P_0(1, -1, -1)$，求(1)$\nabla\varphi$ (即φ的梯度(gradient))；(2)$D_{\mathbf{w}}\varphi(P_0)$ (即φ在P_0沿$\mathbf{w}=\mathbf{i}+\mathbf{j}+\mathbf{k}$方向的方向導數)。[104聯合電子7]

[解](1)$\nabla\varphi=(-2x-yz)\mathbf{i}+(-2y-xz)\mathbf{j}-xy\mathbf{k}$

(2)$\nabla\varphi|_{(1,\,-1,\,-1)}=-3\mathbf{i}+3\mathbf{j}+\mathbf{k}$

$$D_{\mathbf{w}}\varphi(P_0)=\nabla\varphi|_{(1,\,-1,\,-1)}\cdot\frac{\mathbf{w}}{|\mathbf{w}|}=(-3\mathbf{i}+3\mathbf{j}+\mathbf{k})\cdot\frac{\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{1^2+1^2+1^2}}=\frac{1}{\sqrt{3}}$$

Ex. 10

Let $f(x, y)=e^{xy}\sin(x+y)$，(1)In what direction, starting at (0, π/2), is f changing the fastest？(2)In what direction, starting at (0, π/2), is f changing at 50% of its maximum rate？[99中山機電2]

[解](1) f在(0, π/2)變化最快的方向為

$$\nabla f|_{(0,\,\pi/2)}=\left.\left(\frac{\partial f}{\partial x}\mathbf{i}+\frac{\partial f}{\partial y}\mathbf{j}\right)\right|_{(0,\,\pi/2)}$$

$$=e^{xy}\{[y\sin(x+y)+\cos(x+y)]\mathbf{i}+[x\sin(x+y)+\cos(x+y)]\mathbf{j}\}\big|_{(0,\,\pi/2)}=\frac{\pi}{2}\mathbf{i}$$

(2) f在(0, π/2)變化最快為x軸的方向，f在(0, π/2)變化為最快的50%與x軸夾60°或120°，方向為$\pm(\mathbf{i}+\sqrt{3}\mathbf{j})$或$\pm(-\mathbf{i}+\sqrt{3}\mathbf{j})$

[Exercise]1. 已知$f(x, y, z)=2x-y^2+z^2$，(1)求∇f；(2)求由坐標(4, –4, 2)處指向原點方向的單位向量$\mathbf{u}$；(3)求往上述$\mathbf{u}$方向之directional derivative $D_u f(4, -4, 2)$。[104高應大機械甲丙6]

2. Please find (1)the tangent plane to the surface $z=x^2+y^2$ at the point (2, –2, 8). (2)the line normal to the surface $z=x^2+y^2$ at the point (2, –2, 8). [104雲科大機械3]

3. 已知純量函數$\phi(x, y, z)=2xz+e^y z^2$；試問此場在點(1, 0, 1)沿方向$2\mathbf{i}+3\mathbf{j}-\mathbf{k}$之變化率。[97交大土木丁12]

4. 試求在點(1, 1, 1)與曲面$x^2+y^2+z^2=3$相垂直之單位向量，並求出曲面在此點之切平面方程式。[97交大土木丁11]

5. Experiments show that in a temperature field $T=x^3-3xy^2$, heat flows in the direction of maximum decrease of temperature T. Find this direction in general and at a given point P($\sqrt{8}, \sqrt{2}$). [98中山機電]

6. Find the directional derivative of $f=x^2+y^2$ at the point P(1, 1) in the direction of the vector $\mathbf{a}=2\mathbf{i}-6\mathbf{j}$. [104中正光機電整合]

7. Find the directional derivative of $F(x, y, z)=xy^2-4x^2y+z^2$ at (1, –1, 2) in the direction of $6\mathbf{i}+2\mathbf{j}+3\mathbf{k}$. [104中原機械甲6]

[Answer]1. (1)$(2\mathbf{i}-2y\mathbf{j}+2z\mathbf{k})$ (2)$(-2\mathbf{i}+2\mathbf{j}-\mathbf{k})/3$ (3)8/3

2. (1)$4x-4y-z=8$ (2)$\dfrac{x-2}{4}=\dfrac{y+2}{-4}=\dfrac{z-8}{-1}$

3.[解]令 $\mathbf{v}=2\mathbf{i}+3\mathbf{j}-\mathbf{k} \Rightarrow \dfrac{\mathbf{v}}{|\mathbf{v}|}=\dfrac{2\mathbf{i}+3\mathbf{j}-\mathbf{k}}{\sqrt{2^2+3^2+(-1)^2}}=\dfrac{2\mathbf{i}+3\mathbf{j}-\mathbf{k}}{\sqrt{14}}$

$\nabla\phi=2z\mathbf{i}+e^y z^2\mathbf{j}+(2x+2e^y z)\mathbf{k}$

ϕ在點$(1, 0, 1)$沿方向$2\mathbf{i}+3\mathbf{j}-\mathbf{k}$的變化率為

$$\left.(\nabla\phi\cdot\frac{\mathbf{v}}{|\mathbf{v}|})\right|_{(1,\,0,\,1)}=[2\mathbf{i}+\mathbf{j}+(2+2)\mathbf{k}]\cdot\frac{2\mathbf{i}+3\mathbf{j}-\mathbf{k}}{\sqrt{14}}=\frac{3}{\sqrt{14}}$$

4.[解]令 $\phi=x^2+y^2+z^2 \Rightarrow \nabla\phi=2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k}$

在點$P(1, 1, 1)$與曲面垂直的單位向量為

$$\mathbf{n}=\pm\left.\frac{\nabla\phi}{|\nabla\phi|}\right|_{(1,\,1,\,1)}=\pm\frac{2\mathbf{i}+2\mathbf{j}+2\mathbf{k}}{\sqrt{2^2+2^2+2^2}}=\pm\frac{\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{3}}$$

設過點P的切平面上任一點為$X(x, y, z)$，則

$$\mathbf{n}\cdot\overrightarrow{PX}=0\Rightarrow\pm\frac{\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{3}}\cdot[(x-1)\mathbf{i}+(y-1)\mathbf{j}+(z-1)\mathbf{k}]=0$$

得曲面在P的切平面方程式為

$(x-1)+(y-1)+(z-1)=0\Rightarrow x+y+z=3$

5.[解]方向為

$$\nabla T = \frac{\partial T}{\partial x}\mathbf{i} + \frac{\partial T}{\partial y}\mathbf{j} = (3x^2 - 3y^2)\mathbf{i} - 6xy\mathbf{j}$$

在點P($\sqrt{8}, \sqrt{2}$)的方向為

$$(3 \cdot 8 - 3 \cdot 2)\mathbf{i} - 6 \cdot \sqrt{8} \cdot \sqrt{2}\mathbf{j} = 18\mathbf{i} - 24\mathbf{j}$$

6. $-4/\sqrt{10}$　　7. 6

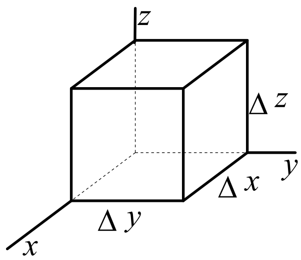
IV. 向量場的散度(Divergence)

定義：設 $\mathbf{v}(x, y, z)$是一個可微分的向量函數，若 $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$，則 $\mathbf{v}$ 的散度為

$$\operatorname{div} \mathbf{v} = \nabla \cdot \mathbf{v} = (\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}) \cdot (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

對一個邊長為Δx, Δy, and Δz 的長方體，如圖所示，$\mathbf{v}$ 在右手邊的面沿著 y 方向流出的量為$v_2(x, y + \Delta y, z)\Delta x\Delta z$，在左手邊的面沿著$y$方向流出的量為 $-v_2(x, y, z)\Delta x\Delta z$，沿著 y 方向淨流出的量為

$$[v_2(x, y + \Delta y, z) - v_2(x, y, z)]\Delta x\Delta z$$



中括號 v_2 的差值為

$$\frac{\partial v_2}{\partial y}\Delta y$$

因此，在 y 方向兩平面淨流出的量為

$$\frac{\partial v_2}{\partial y}\Delta x\Delta y\Delta z$$

同理，在 x 及 z 方向淨流出的量分別為

$$\frac{\partial v_1}{\partial x}\Delta x\Delta y\Delta z \quad 及 \quad \frac{\partial v_3}{\partial z}\Delta x\Delta y\Delta z$$

將這個結果相加後，除以體積$\Delta x\Delta y\Delta z$，得到

$$\nabla \cdot \mathbf{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

為單位體積淨流出的量。

Ex. 11

Write down the definition of divergence (div). Find the div of the given vector function $\mathbf{v}=(z-y)\mathbf{i}+(x-z)\mathbf{j}+(y-x)\mathbf{k}$. [104高應大機械乙5(1)]

[解]若$\mathbf{v}=v_1\mathbf{i}+v_2\mathbf{j}+v_3\mathbf{k}$，則div $\mathbf{v}=\nabla\cdot\mathbf{v}=\dfrac{\partial v_1}{\partial x}+\dfrac{\partial v_2}{\partial y}+\dfrac{\partial v_3}{\partial z}$

此題$v_1=z-y,\ v_2=x-z,\ v_3=y-x$，因此$\nabla\cdot\mathbf{v}=0$

Ex. 12

Find $\nabla\cdot\mathbf{r}$, given $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$.

[解]$\mathbf{r}=r_1\mathbf{i}+r_2\mathbf{j}+r_3\mathbf{k}$，則div $\mathbf{r}=\nabla\cdot\mathbf{r}=\dfrac{\partial r_1}{\partial x}+\dfrac{\partial r_2}{\partial y}+\dfrac{\partial r_3}{\partial z}$

而$r_1=x,\ r_2=y,\ r_3=z$，因此$\nabla\cdot\mathbf{r}=1+1+1=3$

[Exercise]1. If $\mathbf{A}=3xz^2\mathbf{i}-yz\mathbf{j}+(x+2z)\mathbf{k}$, find $\nabla\cdot\mathbf{A}$. [98屏教大光電2(2)]

2. 一3-D向量場為$\mathbf{F}=-2x\mathbf{i}-ze^x\mathbf{j}+(2z-1)\mathbf{k}$，試求$\mathbf{F}$之divergence $\nabla\cdot\mathbf{F}$。[97台科營建3(1)]

3. 已知向量場為$\mathbf{F}=x^2z\mathbf{i}-2y^3z^2\mathbf{j}+xy^2z\mathbf{k}$，求$\mathbf{F}$在點P(1, −1, 1)處的散度$\nabla\cdot\mathbf{F}$。[102勤益機械5(1)]

[Answer]1. $3z^2-z+2$　2. 0　3. −3

V. 向量場的旋度(Curl)

定義：設 $\mathbf{v}(x, y, z) = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ 是一個可微分的向量函數，則 $\mathbf{v}$ 的旋度或 $\mathbf{v}$ 的旋轉為

$$\text{curl}\,\mathbf{v} = \nabla \times \mathbf{v} = (\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}) \times (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= (\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z})\mathbf{i} + (\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x})\mathbf{j} + (\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y})\mathbf{k}$$

Ex. 13

Write down the definition of curl. Find the curl of the given vector function $\mathbf{v} = (z - y)\mathbf{i} + (x - z)\mathbf{j} + (y - x)\mathbf{k}$. [104高應大機械乙5(2)]

[解]設 $\mathbf{v}(x, y, z) = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$，則

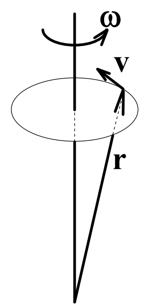
$$\text{curl}\,\mathbf{v} = \nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z-y & x-z & y-x \end{vmatrix} = 2(\mathbf{i} + \mathbf{j} + \mathbf{k})$$

Ex. 14

Suppose a body rotates with a constant angular velocity $\boldsymbol{\omega}$ about an axis. If $\mathbf{r}$ is the position vector of a point P on the body measured from the origin, then the linear velocity vector $\mathbf{v}$ of rotation is $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$. See Figure. If $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $\boldsymbol{\omega} = \omega_1\mathbf{i} + \omega_2\mathbf{j} + \omega_3\mathbf{k}$, show that $\boldsymbol{\omega} = \frac{1}{2}\,\text{curl}\,\mathbf{v}$. [98中興材料]

[解] $\mathbf{v} = (\omega_1\mathbf{i} + \omega_2\mathbf{j} + \omega_3\mathbf{k}) \times (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = (\omega_2 z - \omega_3 y)\mathbf{i} + (\omega_3 x - \omega_1 z)\mathbf{j} + (\omega_1 y - \omega_2 x)\mathbf{k}$

$$\nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \omega_2 z - \omega_3 y & \omega_3 x - \omega_1 z & \omega_1 y - \omega_2 x \end{vmatrix} = 2(\omega_1\mathbf{i} + \omega_2\mathbf{j} + \omega_3\mathbf{k}) = 2\boldsymbol{\omega}$$



因此，旋轉剛體速度場 $\mathbf{v}$ 的旋度為 $2\boldsymbol{\omega}$。

[Exercise]1. If $\mathbf{A}=3xz^2\mathbf{i}-yz\mathbf{j}+(x+2z)\mathbf{k}$, find $\nabla\times\mathbf{A}$. [98屏教大光電2(1)]

2. 一3-D向量場為$\mathbf{F}=-2x\mathbf{i}-ze^x\mathbf{j}+(2z-1)\mathbf{k}$，試求$\mathbf{F}$之curl $\nabla\times\mathbf{F}$。[97台科營建3(2)]

3. 已知向量場為$\mathbf{F}=x^2z\mathbf{i}-2y^3z^2\mathbf{j}+xy^2z\mathbf{k}$，求$\mathbf{F}$在點P(1, −1, 1)處的旋度$\nabla\times\mathbf{F}$。[102勤益機械5(2)]

[Answer]1. $y\mathbf{i}+(6xz-1)\mathbf{j}$　　2. $e^x\mathbf{i}-ze^x\mathbf{k}$　　3. $-6\mathbf{i}$

第三章 向量積分

I. 線積分(Line Integral)

1. 定義

一個向量函數**F**(**r**)在曲線C上的線積分為

$$\int_C \mathbf{F}(\mathbf{r})\cdot d\mathbf{r}$$

以分量表示$\mathbf{F}=F_1\mathbf{i}+F_2\mathbf{j}+F_3\mathbf{k}$， $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}\Rightarrow d\mathbf{r}=dx\mathbf{i}+dy\mathbf{j}+dz\mathbf{k}$，則

$$\int_C \mathbf{F}(\mathbf{r})\cdot d\mathbf{r}=\int_C (F_1\mathbf{i}+F_2\mathbf{j}+F_3\mathbf{k})\cdot(dx\mathbf{i}+dy\mathbf{j}+dz\mathbf{k})=\int_C F_1dx+F_2dy+F_3dz$$

若x、y及z為t的函數，$d\mathbf{r}=(\frac{dx}{dt}\mathbf{i}+\frac{dy}{dt}\mathbf{j}+\frac{dz}{dt}\mathbf{k})dt$ ，則

$$\int_C \mathbf{F}(\mathbf{r})\cdot d\mathbf{r}=\int_C (F_1\mathbf{i}+F_2\mathbf{j}+F_3\mathbf{k})\cdot(\frac{dx}{dt}\mathbf{i}+\frac{dy}{dt}\mathbf{j}+\frac{dz}{dt}\mathbf{k})dt=\int_C (F_1\frac{dx}{dt}+F_2\frac{dy}{dt}+F_3\frac{dz}{dt})dt$$

Ex. 1

若力場$\mathbf{F}=2y\mathbf{i}-x\mathbf{j}$且圓C的圓心為(1, 3)，半徑為2，計算以力場**F**推動一粒子沿圓C正位向繞一圈所做的功$\int_C \mathbf{F}\cdot d\mathbf{r}$。[104聯合電子8(b)]

[解]圓的參數式為$x=1+2\cos t,\ y=3+2\sin t$，因此

$$\mathbf{r}=x\mathbf{i}+y\mathbf{j}=(1+2\cos t)\mathbf{i}+(3+2\sin t)\mathbf{j}\Rightarrow d\mathbf{r}=(-2\sin t\mathbf{i}+2\cos t\mathbf{j})dt$$

$$\mathbf{F}=2y\mathbf{i}-x\mathbf{j}=2(3+2\sin t)\mathbf{i}-(1+2\cos t)\mathbf{j}$$

$$\int_C \mathbf{F}\cdot d\mathbf{r}=\int_0^{2\pi}[2(3+2\sin t)\mathbf{i}-(1+2\cot)\mathbf{j}]\cdot(-2\sin t\mathbf{i}+2\cos t\mathbf{j})dt$$

$$=\int_0^{2\pi}(-12\sin t-8\sin^2 t-2\cos t-4\cos^2 t)dt$$

$$=\int_0^{2\pi}(-12\sin t-4\sin^2 t-2\cos t-4)dt$$

$$=\int_0^{2\pi}[-12\sin t-4\cdot\frac{1-\cos 2t}{2}-2\cos t-4]dt$$

$$=[12\cos t-(2t-\sin 2t)-2\sin t-4t]\Big|_0^{2\pi}=-12\pi$$

Ex. 2

A force field **F** in 3-space is given $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + (xz - y)\mathbf{k}$. Compute the work done by this force in moving a particle from (0, 0, 0) to (1, 2, 4) along the line segment joining these two points. [103北科大化工6]

[解]連接兩點的直線方程式為$\dfrac{x}{1} = \dfrac{y}{2} = \dfrac{z}{4} \Rightarrow$令$x = t,\ y = 2t,\ z = 4t$

$$\mathbf{F} = x\mathbf{i} + y\mathbf{j} + (xz - y)\mathbf{k} = t\mathbf{i} + 2t\mathbf{j} + (t \cdot 4t - 2t)\mathbf{k} = t\mathbf{i} + 2t\mathbf{j} + (4t^2 - 2t)\mathbf{k}$$

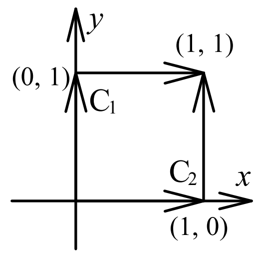
$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \Rightarrow d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k} = dt\mathbf{i} + 2dt\mathbf{j} + 4dt\mathbf{k} = (\mathbf{i} + 2\mathbf{j} + 4\mathbf{k})dt$$

作功為$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C [t\mathbf{i} + 2t\mathbf{j} + (4t^2 - 2t)\mathbf{k}] \cdot (\mathbf{i} + 2\mathbf{j} + 4\mathbf{k})dt = \int_0^1 [t + 4t + 4(4t^2 - 2t)]dt$

$$= \int_0^1 (16t^2 - 3t)dt = \frac{16}{3}t^3 - \frac{3}{2}t^2 \Big|_0^1 = \frac{16}{3} - \frac{3}{2} = \frac{23}{6}$$

Ex. 3

(1)Use the path (C_1) and (2)Use the path (C_2) in Fig. to evaluate the integral $\int_O^P r^2 d\mathbf{r}$, where $r^2 = x^2 + y^2$. [102勤益電子7]



[解](1)(0, 0)到(0, 1) : $x = 0,\ dx = 0,\ r^2 = y^2,\ d\mathbf{r} = dy\mathbf{j}$

$$\int_O^A r^2 d\mathbf{r} = \int_0^1 y^2 dy\mathbf{j} = \frac{y^3}{3}\Big|_0^1 \mathbf{j} = \frac{1}{3}\mathbf{j}$$

(0, 1)到(1, 1) : $y = 1,\ dy = 0,\ r^2 = x^2 + 1,\ d\mathbf{r} = dx\mathbf{i}$

$$\int_A^P r^2 d\mathbf{r} = \int_0^1 (x^2 + 1)dx\mathbf{i} = (\frac{x^3}{3} + x)\Big|_0^1 \mathbf{i} = \frac{4}{3}\mathbf{i}$$

$$\int_O^P r^2 d\mathbf{r} = \int_O^A r^2 d\mathbf{r} + \int_A^P r^2 d\mathbf{r} = \frac{1}{3}\mathbf{j} + \frac{4}{3}\mathbf{i}$$

(2)(0, 0)到(1, 0) : $y = 0,\ dy = 0,\ r^2 = x^2,\ d\mathbf{r} = dx\mathbf{i}$

$$\int_O^B r^2 d\mathbf{r} = \int_0^1 x^2 dx\mathbf{i} = \frac{x^3}{3}\Big|_0^1 \mathbf{i} = \frac{1}{3}\mathbf{i}$$

(1, 0)到(1, 1) : $x = 1,\ dx = 0,\ r^2 = 1 + y^2,\ d\mathbf{r} = dy\mathbf{j}$

$$\int_B^P r^2 d\mathbf{r} = \int_0^1 (1 + y^2)dy\mathbf{j} = (y + \frac{y^3}{3})\Big|_0^1 \mathbf{j} = \frac{4}{3}\mathbf{j}$$

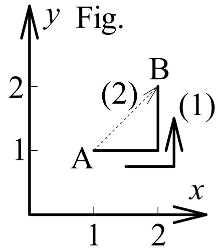
$$\int_O^P r^2 d\mathbf{r} = \int_O^B r^2 d\mathbf{r} + \int_B^P r^2 d\mathbf{r} = \frac{1}{3}\mathbf{i} + \frac{4}{3}\mathbf{j}$$

Ex. 4

Please evaluate the integral $\int_C xy^3 ds$ where C is the segment of the line $y=2x$ in the x-y plane from $(-1,-2)$ to $(1, 2)$. [98交大機械丙4]

[解] $\mathbf{r}=x\mathbf{i}+y\mathbf{j} \Rightarrow d\mathbf{r}=dx\mathbf{i}+dy\mathbf{j}=dx\mathbf{i}+2dx\mathbf{j} \Rightarrow ds=|d\mathbf{r}|=\sqrt{(dx)^2+(2dx)^2}=\sqrt{5}dx$

$$\int_C xy^3 ds=\int_C x(2x)^3\sqrt{5}dx=8\sqrt{5}\int_{-1}^{1}x^4 dx=8\sqrt{5}\cdot\frac{x^5}{5}\bigg|_{-1}^{1}=\frac{16\sqrt{5}}{5}$$



[Exercise]1. Cacualte the line integral of function $\mathbf{v}=y^2x\mathbf{i}+2x(y+1)\mathbf{j}$ from the point A(1,1,0) to the point B(2, 2,0), along the path (1) and (2) in Fig. What is the line integral $\oint \vec{v}\cdot d\vec{l}$ for the loop that goes from A to B along (1) and return to A along (2)? [98屏教大光電10]

2. 已知平面向量場$\mathbf{F}(x, y)=\langle 1, 2\rangle$，求線積分$\int_C \mathbf{F}\cdot d\mathbf{r}$之值，其中曲線C為$\mathbf{r}(t)=\langle 2\cos t, \sin t\rangle$，$0\le t\le 2\pi$。 [104海洋輪機6]

3. 某質點受外力$\mathbf{F}=y\mathbf{i}+2x\mathbf{j}$作用沿以原點為圓心半徑為1之圓弧，自(1, 0)移動至(0, 1)，試計算其所做之功。[97交大土木丁13]

4. Find the work ($\int_C \mathbf{F}\cdot d\mathbf{r}$) done by the force $\mathbf{F}(x, y, z)=e^x\mathbf{i}+xe^{xy}\mathbf{j}+xye^{xyz}\mathbf{k}$ acting along the smooth curve $\mathbf{r}=t\mathbf{i}+t^2\mathbf{j}+t^3\mathbf{k}$, $0\le t\le 1$. [104中興機械5]

5. Please find the work required to move a particle by force $\mathbf{F}(x^2\mathbf{i}-yz\mathbf{j}+x\cos z\mathbf{k})$ along the path C($x=t^2$, $y=t$, $z=\pi t$, $0\le t\le 3$) from point Q(9, 3, 3π) to point P(0, 0, 0) by calculate the integral $-\int_C \mathbf{F}\cdot d\mathbf{R}=-\int_C F_x dx+F_y dy+F_z dz$.

[104中央機械甲乙丙能源光機電乙6; 機械丁光機電甲7]

6. Find the line integral $\int_C zdx+xdy+ydz$, where 2C is the triangle with vertices (3, 0, 0), (0, 0, 2), (0, 6, 0) traversed in the given order. [99宜蘭電機6]

7. Find the integral $\int_{(2,0,1)}^{(4,4,0)} 2x(y^3-z^3)dx+3x^2y^2dy-3x^2z^2dz$. [98宜蘭生物機電1]

[Solution]1. 23/2，137/12 ，1/12 2. 0 3. π/4 4. $13(e-1)/6$ 5. $9\pi+6/\pi-243$ 6. −18 7. 1028

2. 保守場(Conservative Field)

一向量場 $\mathbf{F}=F_1\mathbf{i}+F_2\mathbf{j}+F_3\mathbf{k}$ 若存在一純量函數 ϕ 使得

$$\mathbf{F}=\nabla\phi \Rightarrow F_1\mathbf{i}+F_2\mathbf{j}+F_3\mathbf{k}=\frac{\partial\phi}{\partial x}\mathbf{i}+\frac{\partial\phi}{\partial y}\mathbf{j}+\frac{\partial\phi}{\partial z}\mathbf{k}$$

即 $F_1=\dfrac{\partial\phi}{\partial x}, F_2=\dfrac{\partial\phi}{\partial y}, F_3=\dfrac{\partial\phi}{\partial z}$，$\phi$ 稱為 $\mathbf{F}$ 的位能函數或簡單地稱為位能，此向量場稱為保守場。又

$$\mathbf{F}\cdot d\mathbf{r}=\nabla\phi\cdot d\mathbf{r}=\left(\frac{\partial\phi}{\partial x}\mathbf{i}+\frac{\partial\phi}{\partial y}\mathbf{j}+\frac{\partial\phi}{\partial z}\mathbf{k}\right)\cdot(dx\mathbf{i}+dy\mathbf{j}+dz\mathbf{k})=\frac{\partial\phi}{\partial x}dx+\frac{\partial\phi}{\partial y}dy+\frac{\partial\phi}{\partial z}dz=d\phi$$

沿著曲線 C 由 A 點到 B 點的線積分為

$$\int_{\mathrm{C}}\mathbf{F}\cdot d\mathbf{r}=\int_{\mathrm{A}}^{\mathrm{B}}d\phi=\phi(\mathrm{B})-\phi(\mathrm{A})$$

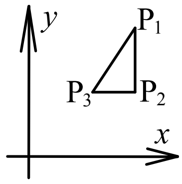
這個結果證明線積分的值只是為路徑 C 兩端點 ϕ 值的差，與路徑無關，因此，在保守力場中，封閉曲線的線積分值為零。又

$$\nabla\times(\nabla\phi)=0 \Rightarrow \nabla\times\mathbf{F}=0$$

因此知道：若且為若 $\nabla\times\mathbf{F}=0$，向量場 $\mathbf{F}$ 是保守場。

Ex. 5

Given a vector function $\mathbf{F}=xy\mathbf{i}+(3x-y^2)\mathbf{j}$, and the coordinates of three points $P_1(5, 6)$, $P_2(5, 3)$, $P_3(3, 3)$, where **i** and **j** are unit vectors along x- and y- axes respectively. (1)Evaluate the integral $\int \mathbf{F}\cdot d\mathbf{r}$ from P_1 straight to P_3. (2)Evaluate the integral $\int \mathbf{F}\cdot d\mathbf{r}$ from P_1 to P_3 along the piecewise straight path $P_1P_2P_3$, i.e., integrate from P_1 along straight line segment to P_2 and then along another straight line from P_2 to P_3. (3)Is this **F** a conservative field? And why? [99 清大動機甲丙丁4]



[解]$\mathbf{F}=xy\mathbf{i}+(3x-y^2)\mathbf{j}\Rightarrow F_1=xy,\ F_2=3x-y^2,\ F_3=0$

(1)連接P_1P_3的直線為$y-6=\dfrac{6-3}{5-3}(x-5)\Rightarrow 3x-2y=3$

設$x=2t+1,\ y=3t\Rightarrow d\mathbf{r}=dx\mathbf{i}+dy\mathbf{j}=(2\mathbf{i}+3\mathbf{j})dt$

$$\mathbf{F}=xy\mathbf{i}+(3x-y^2)\mathbf{j}=(2t+1)\cdot 3t\mathbf{i}+[3(2t+1)-(3t)^2]\mathbf{j}=(6t^2+3t)\mathbf{i}+(-9t^2+6t+3)\mathbf{j}$$

$$\int_C \mathbf{F}\cdot d\mathbf{r}=\int_C [(6t^2+3t)\mathbf{i}+(-9t^2+6t+3)\mathbf{j}]\cdot(2\mathbf{i}+3\mathbf{j})dt=\int_2^1[2(6t^2+3t)+3(-9t^2+6t+3)]dt$$

$$=\int_2^1(-15t^2+24t+9)dt=(-5t^3+12t^2+9t)\Big|_2^1=-10$$

(2)C_1: 連接P_1P_2的線段$x=5,\ dx=0,\ d\mathbf{r}=dy\mathbf{j}$

$$\mathbf{F}=xy\mathbf{i}+(3x-y^2)\mathbf{j}=5y\mathbf{i}+(15-y^2)\mathbf{j}$$

$$\int_{C_1}\mathbf{F}\cdot d\mathbf{r}=\int_{C_1}[5y\mathbf{i}+(15-y^2)\mathbf{j}]\cdot dy\mathbf{j}=\int_6^3(15-y^2)dy=(15y-\frac{y^3}{3})\Big|_6^3=18$$

C_2: 連接P_2P_3的線段$y=3,\ dy=0,\ d\mathbf{r}=dx\mathbf{i}$

$$\mathbf{F}=xy\mathbf{i}+(3x-y^2)\mathbf{j}=3x\mathbf{i}+(3x-9)\mathbf{j}$$

$$\int_{C_2}\mathbf{F}\cdot d\mathbf{r}=\int_{C_2}[3x\mathbf{i}+(3x-9)\mathbf{j}]\cdot dx\mathbf{i}=\int_5^3 3xdx=\frac{3x^2}{2})\Big|_5^3=-24$$

$$\int_C \mathbf{F}\cdot d\mathbf{r}=\int_{C_1}\mathbf{F}\cdot d\mathbf{r}+\int_{C_2}\mathbf{F}\cdot d\mathbf{r}=18+(-24)=-6$$

(3)**F**不是保守場，因為線積分值與路徑有關。

Ex. 6

Show that the differential form under the integral sign of $I=\int_{(-1,5)}^{(4,3)}(3z^2dx+6xzdz)$ is exact, so that we have independence of path in any domain, and find the value of the integral I from A(−1, 5) to B(4, 3). [97中央機械丁光電甲4]

[解]令$F_1=3z^2, F_2=6xz \Rightarrow \frac{\partial F_1}{\partial z}=6z, \frac{\partial F_2}{\partial x}=6z$

$\frac{\partial F_1}{\partial z}=\frac{\partial F_2}{\partial x} \Rightarrow 3z^2dx+6xzdz$為正合

設$\phi(x,z)$滿足$\frac{\partial \phi}{\partial x}=F_1, \frac{\partial \phi}{\partial z}=F_2$，則

$\phi=\int_x F_1dx+f(z)=\int_x 3z^2dx+f(z)=3xz^2+f(z)$

$\phi=\int_z F_2dz+g(x)=\int_z 6xzdz+g(x)=3xz^2+g(x)$

比較兩式，得$\phi(x,z)=3xz^2$

$I=\int_{(-1,5)}^{(4,3)}(3z^2dx+6xzdz)=\phi(x,z)\Big|_{(-1,5)}^{(4,3)}=3\cdot4\cdot3^2-3\cdot(-1)\cdot5^2=183$

[Exercise]1. 已知線積分$\int 2xydx + x^2dy$，積分路徑：$y = 3x-2,\ 1 \le x \le 2$。(1)證明此積分是否與路徑有關？(2)求積分值。[104高應大機械甲丙7]

2. 若力場$\mathbf{F}=2y\mathbf{i}-x\mathbf{j}$且圓C的圓心為(1, 3)，半徑為2，判斷$\mathbf{F}$是否為保守力場。[104聯合電子8(a)]

3. Is the integration $\int (3x^2+3y-1)dx + (z^2+3x)dy + (2yz+1)dz$ independent of the path? [98屏教大光電1]

1. [解]設$\mathbf{F} = 2xy\mathbf{i} + x^2\mathbf{j} \Rightarrow F_1 = 2xy, F_2 = x^2, F_3 = 0$，則該線積分為$\int \mathbf{F}\cdot d\mathbf{r}$

$$(1)\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xy & x^2 & 0 \end{vmatrix} = 2x\mathbf{k} - 2x\mathbf{k} = 0 \Rightarrow \text{此線積分與路徑無關}$$

(2)存在一純量函數ϕ，使得$\nabla\phi = \mathbf{F} \Rightarrow \dfrac{\partial\phi}{\partial x}\mathbf{i} + \dfrac{\partial\phi}{\partial y}\mathbf{j} + \dfrac{\partial\phi}{\partial z}\mathbf{k} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$

$$\frac{\partial\phi}{\partial x} = F_1 \Rightarrow \phi = \int_x F_1 dx = \int_x 2xydx = x^2y + f(y,z)$$

$$\frac{\partial\phi}{\partial y} = F_2 \Rightarrow x^2 + \frac{\partial f}{\partial y} = x^2 \Rightarrow \frac{\partial f}{\partial y} = 0 \Rightarrow f = g(z)$$

$$\frac{\partial\phi}{\partial z} = F_3 \Rightarrow \frac{dg}{dz} = 0 \Rightarrow\ g(z) = C$$

得$\phi = x^2y + C$，線積分從A(1, 1)到B(2, 4)

線積分$\int \mathbf{F}\cdot d\mathbf{r} = \int_{\mathrm{A}}^{\mathrm{B}} d\phi = \phi(\mathrm{A}) - \phi(\mathrm{B}) = 1^2\cdot 1 - 2^2\cdot 4 = 1 - 16 = -15$

2. [解]$\mathbf{F} = 2y\mathbf{i} - x\mathbf{j} \Rightarrow F_1 = 2y, F_2 = -x, F_3 = 0$

$$\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2y & -x & 0 \end{vmatrix} = -\mathbf{k} - 2\mathbf{k} \ne 0 \Rightarrow \mathbf{F}\text{不是保守力場}$$

3. [解]$\mathbf{F} = (3x^2+3y-1)\mathbf{i} + (z^2+3x)\mathbf{j} + (2yz+1)k) \Rightarrow F_1 = 3x^2+3y-1, F_2 = z^2+3x, F_3 = 2yz+1$

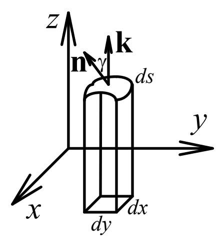
$$\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 3x^2+3y-1 & z^2+3x & 2yz+1 \end{vmatrix} = 0 \Rightarrow \text{積分值與路徑無關}$$

II. 面積分(Surface Integral)

定義：**F** 通過曲面 S 的通量為面積分

$$\iint_S \mathbf{F}\cdot d\mathbf{s} \quad \text{or} \quad \iint_S \mathbf{F}\cdot\mathbf{n}\,ds,$$

其中 **n** 為垂直 $d\mathbf{s}$ 的單位向量。
由圖知 $ds\,|\cos\gamma| = dxdy$，γ 為 $d\mathbf{s}$ 的法向量與 z 軸的夾角，又



$$\mathbf{n}\cdot\mathbf{k} = \cos\gamma \Rightarrow ds = \frac{dxdy}{|\mathbf{n}\cdot\mathbf{k}|}$$

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,ds = \iint_S \mathbf{F}\cdot\mathbf{n}\frac{dxdy}{|\mathbf{n}\cdot\mathbf{k}|}$$

同理，若 α 及 β 分別為 $d\mathbf{s}$ 的法向量與 x 及 y 軸的夾角，則

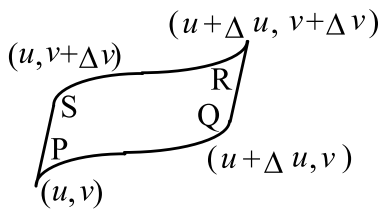
$$\iint_S \mathbf{F}\cdot\mathbf{n}\,ds = \iint_S \mathbf{F}\cdot\mathbf{n}\frac{dydz}{|\mathbf{n}\cdot\mathbf{i}|} = \iint_S \mathbf{F}\cdot\mathbf{n}\frac{dzdx}{|\mathbf{n}\cdot\mathbf{j}|}$$

若以分量表示 $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$, $\mathbf{n} = \cos\alpha\mathbf{i} + \cos\beta\mathbf{j} + \cos\gamma\mathbf{k}$，則

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,ds = \iint_S (F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k})\cdot(\cos\alpha\mathbf{i} + \cos\beta\mathbf{j} + \cos\gamma\mathbf{k})\,ds$$
$$= \iint_S (F_1\cos\alpha + F_2\cos\beta + F_3\cos\gamma)\,ds = \iint_S (F_1 dydz + F_2 dzdx + F_3 dxdy)$$

此時就可以利用**F**的各個分量分別對S投影在三個坐標平面的面積作積分

當曲面 S 以參數 $\mathbf{r}(u, v)$ 表示時



$$\overrightarrow{PQ} = \frac{\partial\mathbf{r}}{\partial u}du,\ \overrightarrow{PS} = \frac{\partial\mathbf{r}}{\partial v}dv$$

$$d\mathbf{s} = \overrightarrow{PQ}\times\overrightarrow{PS} = \frac{\partial\mathbf{r}}{\partial u}\times\frac{\partial\mathbf{r}}{\partial v}dudv$$

$$\iint_S \mathbf{F}\cdot d\mathbf{s} = \iint_S \mathbf{F}\cdot\frac{\partial\mathbf{r}}{\partial u}\times\frac{\partial\mathbf{r}}{\partial v}dudv$$

當 $\mathbf{F} = \mathbf{n}$ 時，面積分為曲面本身的面積。

Ex. 7

Given a vector field $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, and a surface defined by S: $x^2 + y^2 + z^2 = 1$, $z \geq 0$. (1)Calculate $\iint_S \mathbf{F} \cdot \mathbf{n}\, ds$, where $\mathbf{n}$ is an outward normal unit vector. (2)If $\mathbf{F} = -\nabla U$, find $U(x, y, z)$.

[104高應大土木3(3)(4)]

[解]$\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \Rightarrow F_1 = x,\ F_2 = y,\ F_3 = z$

(1)設 $f = x^2 + y^2 + z^2$，S的單位法向量為

$$\mathbf{n} = \frac{\nabla f}{|\nabla f|} = \frac{2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}}{\sqrt{(2x)^2 + (2y)^2 + (2z)^2}} = \frac{2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}}{2\sqrt{x^2 + y^2 + z^2}} = \frac{2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}}{2 \cdot 1} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

$$\iint_S \mathbf{F} \cdot \mathbf{n} ds = \iint_S \mathbf{F} \cdot \mathbf{n} \frac{dxdy}{|\mathbf{n} \cdot \mathbf{k}|} = \iint_S (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \cdot (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \frac{dxdy}{|(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \cdot \mathbf{k}|}$$

$$= \iint_S (x^2 + y^2 + z^2) \frac{dxdy}{z} = 4\int_0^1 \int_0^{\sqrt{1-x^2}} \frac{1}{\sqrt{1-(x^2+y^2)}} dydx$$

令 $x = r\cos\theta,\ y = r\sin\theta$，則

$$\iint_S \mathbf{F} \cdot \mathbf{n} ds = 4\int_0^{\pi/2} \int_0^1 \frac{1}{\sqrt{1-r^2}} \left| \frac{\partial(x, y)}{\partial(r, \theta)} \right| drd\theta = 4\int_0^{\pi/2} \int_0^1 \frac{1}{\sqrt{1-r^2}} r\, drd\theta$$

$$= 4\int_0^{\pi/2} -\frac{1}{2} \cdot 2(1-r^2)^{1/2} \Big|_0^1 d\theta = 4\int_0^{\pi/2} d\theta = 4\theta\Big|_0^{\frac{\pi}{2}} = 4 \cdot \frac{\pi}{2} = 2\pi$$

(2)$\mathbf{F} = -\nabla U \Rightarrow F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k} = -(\frac{\partial U}{\partial x}\mathbf{i} + \frac{\partial U}{\partial y}\mathbf{j} + \frac{\partial U}{\partial z}\mathbf{k})$

$$F_1 = -\frac{\partial U}{\partial x} \Rightarrow U = -\int_x F_1 dx = -\int_x x dx = -\frac{x^2}{2} + f(y, z)$$

$$F_2 = -\frac{\partial U}{\partial y} \Rightarrow y = -\frac{\partial f}{\partial y} \Rightarrow f = -\int_y y dy + g(z) = -\frac{y^2}{2} + g(z)$$

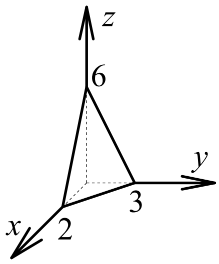
$$F_3 = -\frac{\partial U}{\partial z} \Rightarrow z = -\frac{dg}{dz} \Rightarrow g(z) = -\int z dz + C = -\frac{z^2}{2} + C$$

得 $U = -\frac{x^2}{2} - \frac{y^2}{2} - \frac{z^2}{2} + C$

Ex. 8

Let $\mathbf{F}=z\mathbf{j}+z\mathbf{k}$ represent the flow of a liquid. Find the flux of $\mathbf{F}$ through the surface S given by that portion of the plane $3x+2y+z=6$ in the first octant oriented upward. [104彰師大物理甲光電甲6]

[解]令$f=3x+2y+z$，S的單位法向量為



$$\mathbf{n}=\frac{\nabla f}{|\nabla f|}=\frac{3\mathbf{i}+2\mathbf{j}+\mathbf{k}}{\sqrt{3^2+2^2+1^2}}=\frac{3\mathbf{i}+2\mathbf{j}+\mathbf{k}}{\sqrt{14}}$$

$$\iint_S \mathbf{F}\cdot d\mathbf{s}=\iint_S \mathbf{F}\cdot\mathbf{n}\frac{dxdy}{|\mathbf{n}\cdot\mathbf{k}|}=\iint_S (z\mathbf{j}+z\mathbf{k})\cdot\frac{3\mathbf{i}+2\mathbf{j}+\mathbf{k}}{\sqrt{14}}\frac{dxdy}{\left|\frac{3\mathbf{i}+2\mathbf{j}+\mathbf{k}}{\sqrt{14}}\cdot\mathbf{k}\right|}$$

$$=\iint_S 3zdxdy=\int_0^2\int_0^{3-3x/2}(18-9x-6y)dydx$$

$$=\int_0^2(18y-9xy-3y^2)\Big|_0^{3-3x/2}dx$$

$$=\int_0^2[(54-27x)-(27x-27x^2/2)-(27-27x+27x^2/4)]dx$$

$$=\int_0^2(27x^2/4-27x+27)dx=(9x^3/4-27x^2/2+27x)\Big|_0^2=18$$

[Exercise]1. Calculate the surface integral of the vector function $\mathbf{F}=x\mathbf{i}+y\mathbf{j}$ over the portion of the surface of the unit sphere S: $x^2+y^2+z^2=1$ above the xy-plane, $z\geq 0$.

2. Find the surface integral of the vector function $\mathbf{F}=y^3\mathbf{i}+x^3\mathbf{j}+z^3\mathbf{k}$ over the portion of the surface defined as S: $x^2+4y^2=4,\ x\geq 0,\ y\geq 0,\ 0\leq z\leq h$. [101台南大學電機9]

1. [解]Let $f=x^2+y^2+z^2$，S的單位法向量為

$$\mathbf{n}=\frac{\nabla f}{|\nabla f|}=\frac{2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k}}{\sqrt{(2x)^2+(2y)^2+(2z)^2}}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$$

$$\iint_S \mathbf{F}\cdot d\mathbf{s}=\iint_S \mathbf{F}\cdot\mathbf{n}\frac{dxdy}{|\mathbf{n}\cdot\mathbf{k}|}=\iint_S (x^2+y^2)\frac{dxdy}{z}=4\int_0^1\int_0^{\sqrt{1-x^2}}\frac{x^2+y^2}{\sqrt{1-(x^2+y^2)}}dydx$$

Let $x=r\cos\theta$, $y=r\sin\theta$, we have

$$\iint_S \mathbf{F}\cdot d\mathbf{s}=4\int_0^{\pi/2}\int_0^1\frac{r^2}{\sqrt{1-r^2}}r\,drd\theta=4\int_0^{\pi/2}\int_0^{\pi/2}\frac{\sin^2\phi}{\cos\phi}\cos\phi\sin\phi\,d\phi d\theta$$

$$=4\int_0^{\pi/2}\int_0^{\pi/2}(1-\cos^2\phi)\sin\phi\,d\phi d\theta$$

$$=4\int_0^{\pi/2}\left(-\cos\phi+\frac{\cos^3\phi}{3}\right)_0^{\pi/2}d\theta=4\int_0^{\pi/2}(1-\frac{1}{3})\,d\theta=\frac{4\pi}{3}$$

2. [解]Let $f = x^2 + 4y^2$，S的單位法向量為

$$\mathbf{n} = \frac{\nabla f}{|\nabla f|} = \frac{2x\mathbf{i} + 8y\mathbf{j}}{\sqrt{(2x)^2 + (8y)^2}} = \frac{2x\mathbf{i} + 8y\mathbf{j}}{2\sqrt{x^2 + 16y^2}} = \frac{x\mathbf{i} + 4y\mathbf{j}}{\sqrt{x^2 + 16y^2}}$$

$$\iint_S \mathbf{F} \cdot d\mathbf{s} = \iint_S \mathbf{F} \cdot \mathbf{n} \frac{dydz}{|\mathbf{n} \cdot \mathbf{i}|} = \iint_S (y^3\mathbf{i} + x^3\mathbf{j} + z^3\mathbf{k}) \cdot \frac{x\mathbf{i} + 4y\mathbf{j}}{\sqrt{x^2 + 16y^2}} \frac{dydz}{\left| \frac{x\mathbf{i} + 4y\mathbf{j}}{\sqrt{x^2 + 16y^2}} \cdot \mathbf{i} \right|}$$

$$= \iint_S \frac{xy^3 + 4x^3y}{|x|} dydz = \iint_S \frac{xy^3 + 4x^3y}{x} dydz = \iint_S (y^3 + 4x^2y) dydz$$

$$= \int_0^h \int_0^1 [y^3 + 4(4 - 4y^2)y] dydz = \int_0^h \int_0^1 (-15y^3 + 16y) dydz$$

$$= \int_0^h \left. (-\frac{15}{4}y^4 + 8y^2) \right|_0^1 dz = \int_0^h \frac{17}{4} dz = \frac{17}{4}h$$

III. 體積分(Volume Integral)

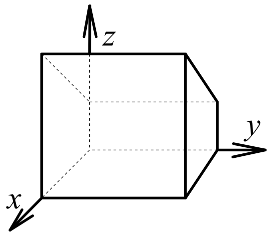
定義：函數f對體積V的體積分為

$$\iiint_V f dv$$

在卡式坐標系統中$dv=dxdydz$，若 $f=1$，則體積分為V的體積。

Ex. 9

Find the volume integral of $f(x, y, z)=x+2yz$ over the box bounded by the coordinate planes, $x=1$, $y=2$, and $z=1+x$.



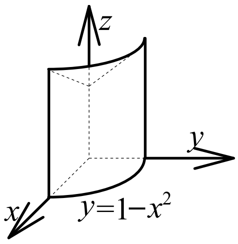
[解]$\int_0^2\int_0^1\int_0^{1+x}(x+2yz)\,dzdxdy=\int_0^2\int_0^1\left(xz+yz^2\right)_0^{1+x}dxdy$

$=\int_0^2\int_0^1[x(1+x)+y(1+x)^2]\,dxdy=\int_0^2\left[(\frac{x^2}{2}+\frac{x^3}{3})+y\frac{(1+x)^3}{3}\right]_0^1 dy$

$=\int_0^2(\frac{5}{6}+\frac{7}{3}y)\,dy=\frac{5}{6}y+\frac{7}{6}y^2\Big|_0^2=\frac{19}{3}$

Ex. 10

Find the volume of the region of space above the xy plane and beneath the plane $z=2+x+y$, bounded by the planes $y=0$, $x=0$, and the surface $y=1-x^2$.



[解]$\int_0^1\int_0^{1-x^2}\int_0^{2+x+y}dzdydx=\int_0^1\int_0^{1-x^2}(2+x+y)\,dydx$

$=\int_0^1\left[(2+x)y+\frac{y^2}{2}\right]_0^{1-x^2}dx=\int_0^1[(2+x)(1-x^2)+\frac{(1-x^2)^2}{2}]\,dx$

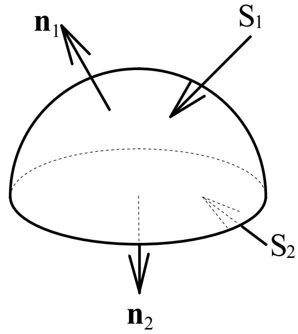
$=\int_0^1(\frac{x^4}{2}-x^3-3x^2+x+\frac{5}{2})\,dx=\left(\frac{x^5}{10}-\frac{x^4}{4}-x^3+\frac{x^2}{2}+\frac{5}{2}x\right)_0^1$

$=\frac{1}{10}-\frac{1}{4}-1+\frac{1}{2}+\frac{5}{2}=\frac{37}{20}$

IV. 散度定理(高斯定理) (Divergence Theorem, Gauss Theorem)

散度定理：設V為一由分段平滑曲面S所圍成的封閉區域，**F**為在含V的某區間內具有連續一階偏導數的向量函數，則

$$\oiint_{S} \mathbf{F}\cdot d\mathbf{s} = \iiint_{V} \nabla\cdot\mathbf{F}dv.$$



以一個半球來說明散度定理：
封閉區面的面積分 $\oiint_{S} \mathbf{F}\cdot d\mathbf{s}$ 等於對S_1及S_2的面積分和，即

$$\oiint_{S} \mathbf{F}\cdot d\mathbf{s} = \iint_{S_1} \mathbf{F}\cdot d\mathbf{s} + \iint_{S_2} \mathbf{F}\cdot d\mathbf{s} = \iint_{S_1} \mathbf{F}\cdot \mathbf{n}_1 ds + \iint_{S_2} \mathbf{F}\cdot \mathbf{n}_2 ds$$

而$\mathbf{n}_1$與$\mathbf{n}_2$均是向外，因此，求出的面積分表**F**從這兩個面積流出的量，而這兩個面積正好圍成一個體積，因此，也等於從這個體積流出的量。

又前面在定義散度時已說明∇ · **F**表示單位體積流出的量，因此，∇ · **F** 做體積分 $\iiint_{V} \nabla\cdot\mathbf{F}dv$ 即表示從這個體積流出的量，所以兩者是相等的。

Ex. 11

Evaluate the surface integral $\oiint_S (x-z)dydz+(2y-z)dzdx-(2x-y)dxdy$ on the surface of the sphere S: $x^2+y^2+z^2=9$. [102彰師大電機4]

[解](1)Let $f=x^2+y^2+z^2$，S的單位法向量為

$$\mathbf{n}=\frac{\nabla f}{|\nabla f|}=\frac{2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k}}{\sqrt{(2x)^2+(2y)^2+(2z)^2}}=\frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{\sqrt{x^2+y^2+z^2}}=\frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{3}$$

$$\iint_S \mathbf{F}\cdot d\mathbf{s}=\iint_S \mathbf{F}\cdot\mathbf{n}\frac{dxdy}{|\mathbf{n}\cdot\mathbf{k}|}$$

$$=\iint_S [(x-z)\mathbf{i}+(2y-z)\mathbf{j}-(2x-y)\mathbf{k}]\cdot\frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{3}\frac{dxdy}{|\frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{3}\cdot\mathbf{k}|}$$

$$=\iint_S [x(x-z)+y(2y-z)-z(2x-y)]\frac{dxdy}{|z|}=\iint_S (x^2+2y^2-3xz)\frac{dxdy}{|z|}$$

其中$3xz$為x的奇函數$\Rightarrow \iint_S -3xz\frac{dxdy}{|z|}=0$

$$\iint_S \mathbf{F}\cdot d\mathbf{s}=\iint_S \frac{x^2+2y^2}{|z|}dxdy=8\int_0^3\int_0^{\sqrt{9-x^2}}\frac{x^2+2y^2}{\sqrt{9-(x^2+y^2)}}dxdy$$

令$x=r\cos\theta$, $y=r\sin\theta$, 則

$$\iint_S \mathbf{F}\cdot d\mathbf{s}=8\int_0^{\pi/2}\int_0^3\frac{r^2(1+\sin^2\theta)}{\sqrt{9-r^2}}\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right|drd\theta=8\int_0^{\pi/2}\int_0^3\frac{r^2(1+\sin^2\theta)}{\sqrt{9-r^2}}rdrd\theta$$

令$r=3\sin\phi\Rightarrow dr=3\cos\phi d\phi$, $\sqrt{9-r^2}=3\cos\phi$

$$\iint_S \mathbf{F}\cdot d\mathbf{s}=8\int_0^{\pi/2}\int_0^{\pi/2}\frac{(3\sin\phi)^3}{3\cos\phi}(3\cos\phi d\phi)(1+\sin^2\theta)d\theta=8\int_0^{\pi/2}\int_0^{\pi/2}27\sin^3\phi d\phi(1+\sin^2\theta)d\theta$$

$$=8\int_0^{\pi/2}\int_0^{\pi/2}27\sin^2\phi(\sin\phi d\phi)(1+\sin^2\theta)d\theta$$

$$=8\int_0^{\pi/2}\int_0^{\pi/2}-27(1-\cos^2\phi)d(\cos\phi)(1+\sin^2\theta)d\theta$$

$$=8\int_0^{\pi/2}-27(\cos\phi-\frac{\cos^3\phi}{3})\Big|_0^{\pi/2}(1+\sin^2\theta)d\theta=8\cdot18\int_0^{\pi/2}(1+\sin^2\theta)d\theta$$

$$=144\int_0^{\pi/2}(1+\frac{1-\cos2\theta}{2})d\theta=144\int_0^{\pi/2}\frac{3-\cos2\theta}{2}d\theta=144(\frac{3}{2}\theta-\frac{\sin2\theta}{4})\Big|_0^{\pi/2}=108\pi$$

(2)$\mathbf{F}=(x-z)\mathbf{i}+(2y-z)\mathbf{j}-(2x-y)\mathbf{k}\Rightarrow\nabla\cdot\mathbf{F}=3$，由散度定理知

$$\oiint_S \mathbf{F}\cdot d\mathbf{s}=\iiint_V \nabla\cdot\mathbf{F}dv=\iiint_V 3dv=3\cdot(\frac{4}{3}\pi\cdot3^3)=108\pi$$

[Exercises]1. Evaluate the surface integral $\oiint_S 4xdydz - zdxdy$, over the sphere S: $x^2+y^2+z^2=4$.

[101彰師大電機5]

2. Evaluate $\iint_S (7x\mathbf{i} - z\mathbf{k}) \cdot d\mathbf{s}$, with S: $x^2+y^2+z^2=4$.

3. Evaluate $\iint_S [xy\mathbf{i} + xz\mathbf{j} + (1-z-yz)\mathbf{k}] \cdot d\mathbf{s}$, with S : $z = 1 - x^2 - y^2$, $z \geq 0$.

1. 32π

2. [解](1)$\mathbf{n} = (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})/2$

$$\iint_S (7x\mathbf{i} - z\mathbf{k}) \cdot \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{2} \frac{dxdy}{|\mathbf{n} \cdot \mathbf{k}|} = \iint_S \frac{7x^2 - z^2}{z} dxdy$$

$$= \iint_S \frac{7x^2 - (4 - x^2 - y^2)}{\sqrt{4 - (x^2 + y^2)}} dxdy = \iint_S \frac{8x^2 + y^2 - 4}{\sqrt{4 - (x^2 + y^2)}} dxdy$$

$$= 8\int_0^{\pi/2} \int_0^2 \frac{8r^2 - 7r^2 \sin^2 \theta - 4}{\sqrt{4 - r^2}} rdrd\theta$$

$$= 8\int_0^{\pi/2} (\int_0^2 \frac{8r^2 - 7r^2 \sin^2 \theta}{\sqrt{4 - r^2}} rdr - \int_0^2 \frac{4}{\sqrt{4 - r^2}} rdr) d\theta$$

$$= 8\int_0^{\pi/2} \left[\int_0^{\pi/2} \frac{8(2\sin\phi)^3 - 7(2\sin\phi)^3 \sin^2\theta}{2\cos\phi} (2\cos\phi\, d\phi) + 2\sqrt{4 - r^2}\Big|_0^2 \right] d\theta$$

$$= 8\int_0^{\pi/2} \int_0^{\pi/2} (64\sin^3\phi - 56\sin^3\phi \sin^2\theta)\, d\phi\, d\theta - 8\int_0^{\pi/2} 8d\theta$$

$$= 8\int_0^{\pi/2} \int_0^{\pi/2} [(64(1 - \cos^2\phi) - 56(1 - \cos^2\phi)\sin^2\theta](\sin\phi d\phi)\, d\theta - 32\pi$$

$$= 8\int_0^{\pi/2} (128/3 - 112/3 \sin^2\theta)\, d\theta - 32\pi = 64\pi$$

(2)$\nabla \cdot (7x\mathbf{i} - z\mathbf{k}) = 7 - 1 = 6$

$$\iint_S (7x\mathbf{i} - z\mathbf{k}) \cdot d\mathbf{s} = \iiint_V 6dv = 6[\frac{4}{3}\pi(2)^3] = 64\pi$$

3. [解](1) Let $f=x^2+y^2+z$

$$\mathbf{n}=\frac{2x\mathbf{i}+2y\mathbf{j}+\mathbf{k}}{\sqrt{4x^2+4y^2+1}}$$

$$\iint_S \mathbf{F}\cdot\mathbf{n}ds=\iint_S \mathbf{F}\cdot\mathbf{n}\frac{dxdy}{|\mathbf{n}\cdot\mathbf{k}|}=\iint_S (2x^2y+2xyz+1-z-yz)dxdy$$

$$=\iint_S [xy\mathbf{i}+xz\mathbf{j}+(1-z-yz)\mathbf{k}]\cdot\frac{2x\mathbf{i}+2y\mathbf{j}+\mathbf{k}}{\sqrt{4x^2+4y^2+1}}\frac{dxdy}{\left|\frac{2x\mathbf{i}+2y\mathbf{j}+\mathbf{k}}{\sqrt{4x^2+4y^2+1}}\cdot\mathbf{k}\right|}$$

$$=\iint_S (2x^2y+2xyz+1-z-yz)dxdy$$

$$=\iint_S [2x^2y+2xy(1-x^2-y^2)+1-(1-x^2-y^2)-y(1-x^2-y^2)]dxdy$$

$$=\iint_S [3x^2y+2xy(1-x^2-y^2)+(x^2+y^2)-y+y^3]dxdy$$

$$=\int_0^{2\pi}\int_0^1[3r^3\cos^2\theta\sin\theta+2r^2\sin\theta\cos\theta(1-r^2)+r^2-r\sin\theta+r^3\sin^3\theta]rdrd\theta$$

$$=\int_0^{2\pi}(\frac{3}{5}\cos^2\theta\sin\theta+\frac{1}{6}\sin\theta\cos\theta+\frac{1}{4}-\frac{1}{3}\sin\theta+\frac{1}{5}\sin^3\theta)d\theta$$

$$=(-\frac{1}{5}\cos^3\theta-\frac{1}{24}\cos 2\theta+\frac{1}{4}\theta+\frac{1}{3}\cos\theta-\frac{1}{5}\cos\theta+\frac{1}{15}\cos^3\theta)\Big|_0^{2\pi}=\frac{\pi}{2}$$

(2) $$\iint_S \mathbf{F}\cdot\mathbf{n}ds=\iiint_V \nabla\cdot\mathbf{F}dv-\iint_{S_1}\mathbf{F}\cdot\mathbf{n}ds=\iiint_V(-1)dv-\iint_{S_1}\mathbf{F}\cdot(-\mathbf{k})dxdy$$

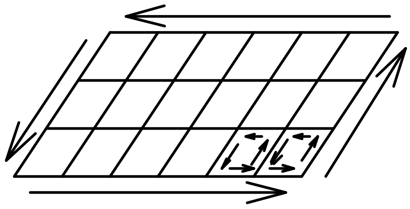
$$=-\int_{-1}^1\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_0^{1-x^2-y^2}dzdxdy+\iint_{S_1}(1-z-yz)\big|_{z=0}dxdy$$

$$=-\int_0^{2\pi}\int_0^1(1-r^2)rdrd\theta+\pi=-\int_0^{2\pi}(\frac{1}{2}-\frac{1}{4})d\theta+\pi=\frac{\pi}{2}$$

V. 史托克斯定理(Stokes Theorem)

史托克斯定理：設S為空間分段平滑的曲面，其邊界為一分段平滑的簡單封閉曲線C，**F**為在含S的某區間內具有連續一階偏導數的向量函數，則

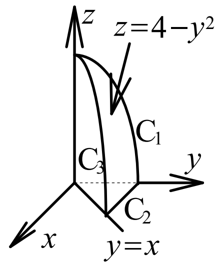
$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{s} = \oint_C \mathbf{F} \cdot d\mathbf{r}$$



[註]C的正向定義為：當你在S的正面上沿著C走，封閉區域必須在你的左方。

Ex. 12

Verify Stoke's theorem, by calculating both sides of the equation, for the case $\mathbf{v} = xz\mathbf{j}$, and S is the surface $z = 4 - y^2$ cut off by the planes $x = 0$, $z = 0$ and $y = x$. [103台科大機械4]



[解](1)令 $f = y^2 + z \Rightarrow \mathbf{n} = \dfrac{\nabla f}{|\nabla f|} = \dfrac{2y\mathbf{j} + \mathbf{k}}{\sqrt{(2y)^2 + 1}} = \dfrac{2y\mathbf{j} + \mathbf{k}}{\sqrt{4y^2 + 1}}$

$$\nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & xz & 0 \end{vmatrix} = -x\mathbf{i} + z\mathbf{k}$$

$$(\nabla \times \mathbf{v}) \cdot \mathbf{n} = \frac{z}{\sqrt{4y^2 + 1}}, \ \mathbf{n} \cdot \mathbf{k} = \frac{1}{\sqrt{4y^2 + 1}}$$

$$\iint_S (\nabla \times \mathbf{v}) \cdot d\mathbf{s} = \iint_S (\nabla \times \mathbf{v}) \cdot \mathbf{n} ds = \int_0^2 \int_0^y (\nabla \times \mathbf{v}) \cdot \mathbf{n} \frac{dxdy}{|\mathbf{n} \cdot \mathbf{k}|}$$

$$= \int_0^2 \int_0^y zdxdy = \int_0^2 \int_0^y (4 - y^2) dxdy = \int_0^2 (4x - xy^2) \Big|_0^y dy$$

$$= \int_0^2 (4y - y^3) dy = (2y^2 - \frac{y^4}{4}) \Big|_0^2 = 4$$

(2)在 C_1 上 $x = 0$、在 C_2 上 $z = 0 \Rightarrow \mathbf{v} = 0$

在 C_3 上 $z = 4 - y^2, \ y = x \Rightarrow \mathbf{v} = xz\mathbf{j} = y(4 - y^2)\mathbf{j}$

$$\oint_C \mathbf{v} \cdot d\mathbf{r} = \int_{C_1} \mathbf{v} \cdot d\mathbf{r} + \int_{C_2} \mathbf{v} \cdot d\mathbf{r} + \int_{C_3} \mathbf{v} \cdot d\mathbf{r}$$

$$= 0 + 0 + \int_0^2 y(4 - y^2)\mathbf{j} \cdot (dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k})$$

$$= \int_0^2 y(4 - y^2) dy = (2y^2 - \frac{y^4}{4}) \Big|_0^2 = 4$$

Ex. 13

Evaluate $\oint_C \mathbf{F}\cdot d\mathbf{r}$, where C is the circle $x^2+y^2=4$, $z=-3$, oriented counterclockwise as seen by a person standing at the origin, and $\mathbf{F}=y\mathbf{i}+xz^3\mathbf{j}-zy^3\mathbf{k}$.

[解](1)令$x=2\cos\theta,\ y=2\sin\theta$

$$\mathbf{F}\cdot d\mathbf{r}=(2\sin\theta\mathbf{i}-54\cos\theta\mathbf{j}+24\sin^3\theta\mathbf{k})\cdot(-2\sin\theta\mathbf{i}+2\cos\theta\mathbf{j})d\theta$$
$$=(-4\sin^2\theta-108\cos^2\theta)d\theta$$
$$\oint_C \mathbf{F}\cdot d\mathbf{r}=\int_0^{2\pi}(-4\sin^2\theta-108\cos^2\theta)\,d\theta=-112\pi$$

(2)設S為圓柱體$x^2+y^2\le 4$被平面$z=-3$所切的面積，它的邊界正好是C；此時$\mathbf{n}=\mathbf{k}$，且

$$\nabla\times\mathbf{F}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ y & xz^3 & -zy^3\end{vmatrix}=(-3y^2z-3xz^2)\mathbf{i}+(z^3-1)\mathbf{k}$$
$$\nabla\times\mathbf{F}\big|_{z=-3}=(9y^2-27x)\mathbf{i}-28\mathbf{k}\Rightarrow(\nabla\times\mathbf{F})\cdot\mathbf{n}=-28$$
$$\iint_S(\nabla\times\mathbf{F})\cdot d\mathbf{s}=\iint_S(\nabla\times\mathbf{F})\cdot\mathbf{n}ds=\iint_S-28\,ds=-28(\pi\cdot 2^2)=-112\pi$$

[Exercise] 1. Find the surface integral $\iint_S(\nabla\times\mathbf{F})\cdot d\mathbf{s}$, with $\mathbf{F}=y\mathbf{i}+z\mathbf{j}+x\mathbf{k}$ and S: the paraboloid $z=1-(x^2+y^2)$, $z\ge 0$.

2. $\mathbf{F}=(2x-y)\mathbf{i}-yz^2\mathbf{j}-y^2z^2\mathbf{k}$, calculate $\iint_S(\nabla\times\mathbf{F})\cdot\mathbf{n}ds$ using Stoke's theorem, where S is the half upper surface of the sphere $x^2+y^2+z^2=1$, $(0\le z\le 1)$. [98 嘉大土木 5]

[解]1. (1)$\mathbf{n}=\dfrac{2x\mathbf{i}+2y\mathbf{j}+\mathbf{k}}{\sqrt{4x^2+4y^2+1}}$, $\nabla\times\mathbf{F}=-\mathbf{i}-\mathbf{j}-\mathbf{k}$

$$(\nabla\times\mathbf{F})\cdot\mathbf{n}=\frac{-2x-2y-1}{\sqrt{4x^2+4y^2+1}},\quad \mathbf{n}\cdot\mathbf{k}=\frac{1}{\sqrt{4x^2+4y^2+1}}$$
$$\iint_S(\nabla\times\mathbf{F})\cdot d\mathbf{s}=\iint(\nabla\times\mathbf{F})\cdot\mathbf{n}\frac{dxdy}{|\mathbf{n}\cdot\mathbf{k}|}=\iint(-2x-2y-1)dxdy$$

其中$-2x,\ -2y$為奇函數$\Rightarrow\iint(-2x-2y)dxdy=0$

$$\iint_S(\nabla\times\mathbf{F})\cdot d\mathbf{s}=\iint(-1)dxdy=-(\pi\cdot 1^2)=-\pi$$

(2) $\iint_S (\nabla\times\mathbf{F})\cdot d\mathbf{s}=\oint_C \mathbf{F}\cdot d\mathbf{r}$, where $C: x^2+y^2=1$

令 $x=\cos\theta,\ y=\sin\theta$

$\mathbf{F}\cdot d\mathbf{r}=(\sin\theta\mathbf{i}+0\mathbf{j}+\cos\theta\mathbf{k})\cdot(-\sin\theta\mathbf{i}+\cos\theta\mathbf{j})d\theta=-\sin^2\theta d\theta$

$$\oint_C \mathbf{F}\cdot d\mathbf{r}=\int_0^{2\pi}-\sin^2\theta\, d\theta=\int_0^{2\pi}-\frac{1-\cos 2\theta}{2}d\theta=-\frac{\theta}{2}+\frac{\sin 2\theta}{4}\Big|_0^{2\pi}=-\pi$$

[解]2. (1)令 $f=x^2+y^2+z^2\Rightarrow \mathbf{n}=\dfrac{2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k}}{\sqrt{(2x)^2+(2y)^2+(2z)^2}}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$

$$\nabla\times\mathbf{F}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ 2x-y & -yz^2 & -y^2z^2\end{vmatrix}=2y(-z^2+z)\mathbf{i}+\mathbf{k}$$

$(\nabla\times\mathbf{F})\cdot\mathbf{n}=2xy(-z^2+z)+z,\ \ \mathbf{n}\cdot\mathbf{k}=z$

$$\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}ds=\iint(\nabla\times\mathbf{F})\cdot\mathbf{n}\frac{dxdy}{|\mathbf{n}\cdot\mathbf{k}|}=\iint[2xy(-z^2+z)+z]\frac{dxdy}{z}$$

其中$2xy(-z^2+z)$為x(或y)的奇函數$\Rightarrow \iint 2xy(-z^2+z)\dfrac{dxdy}{z}=0$

$$\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}ds=\iint dxdy=\pi\cdot 1^2=\pi$$

(2)由Stokes定理知$\iint_S (\nabla\times\mathbf{F})\cdot d\mathbf{s}=\oint_C \mathbf{F}\cdot d\mathbf{r}$, where $C: x^2+y^2=1$

令 $x=\cos\theta,\ y=\sin\theta$

$\mathbf{F}\cdot d\mathbf{r}=(2\cos\theta-\sin\theta)\mathbf{i}+0\mathbf{j}+0\mathbf{k})\cdot(-\sin\theta\mathbf{i}+\cos\theta\mathbf{j})d\theta=(-\sin 2\theta+\sin^2\theta)d\theta$

$$\oint_C \mathbf{F}\cdot d\mathbf{r}=\int_0^{2\pi}(-\sin 2\theta+\sin^2\theta)\,d\theta=\int_0^{2\pi}(-\sin 2\theta+\frac{1-\cos 2\theta}{2})\,d\theta$$

$$=(\frac{\cos 2\theta}{2}+\frac{\theta}{2}-\frac{\sin 2\theta}{4})\Big|_0^{2\pi}=\pi$$

VI. 格林定理(Green's Theorem)

設S為xy平面上單連通封閉區域，S的邊界為分段平滑的簡單封閉曲線C，若$\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j}$ 在此區域內為連續可微分的向量函數，則

$$(\nabla \times \mathbf{F}) \cdot \mathbf{n} = (\nabla \times \mathbf{F}) \cdot \mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & 0 \end{vmatrix} \cdot \mathbf{k} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{s} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{k} ds = \iint_S \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} dx\,dy$$

而

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C F_1 dx + F_2 dy$$

由Stokes定理知

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{s} = \oint_C \mathbf{F} \cdot d\mathbf{r}$$

因此

$$\iint_S \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} dx\,dy = \oint_C F_1 dx + F_2 dy$$

稱為格林定理(Green's theorem)。

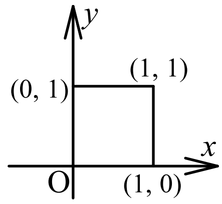
Ex. 14

Evaluate $\oint_C (x^2-y^2)dx+(x^2+y^2)dy$, if C is the square with vertices (0, 0), (1, 0) , (1, 1) and (0, 1). [99台大應力乙6]

[解]$\mathbf{F}=(x^2-y^2)\mathbf{i}+(x^2+y^2)\mathbf{j}\Rightarrow F_1=x^2-y^2,\ F_2=x^2+y^2$

(1)在C_1: $y=0,\ dy=0,\ \mathbf{F}=x^2\mathbf{i}+x^2\mathbf{j},\ d\mathbf{r}=dx\mathbf{i}$

$$\int_{C_1}\mathbf{F}\cdot d\mathbf{r}=\int_{C_1}(x^2\mathbf{i}+x^2\mathbf{j})\cdot(dx\mathbf{i})=\int_0^1 x^2dx=\frac{1}{3}$$



在C_2: $x=1,\ dx=0,\ \mathbf{F}=(1-y^2)\mathbf{i}+(1+y^2)\mathbf{j},\ d\mathbf{r}=dy\mathbf{j}$

$$\int_{C_2}\mathbf{F}\cdot d\mathbf{r}=\int_{C_2}[(1-y^2)\mathbf{i}+(1+y^2)\mathbf{j}]\cdot(dy\mathbf{j})=\int_0^1(1+y^2)dy=\frac{4}{3}$$

在C_3: $y=1,\ dy=0,\ \mathbf{F}=(x^2-1)\mathbf{i}+(x^2+1)\mathbf{j},\ d\mathbf{r}=dx\mathbf{i}$

$$\int_{C_3}\mathbf{F}\cdot d\mathbf{r}=\int_{C_3}[(x^2-1)\mathbf{i}+(x^2+1)\mathbf{j}]\cdot(dx\mathbf{i})=\int_1^0(x^2-1)dx=\frac{2}{3}$$

在C_4: $x=0,\ dx=0,\ \mathbf{F}=-y^2\mathbf{i}+y^2\mathbf{j},\ d\mathbf{r}=dy\mathbf{j}$

$$\int_{C_4}\mathbf{F}\cdot d\mathbf{r}=\int_{C_4}(-y^2\mathbf{i}+y^2\mathbf{j})\cdot(dy\mathbf{j})=\int_1^0 y^2dy=-\frac{1}{3}$$

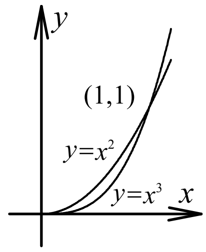
$$\int_C\mathbf{F}\cdot d\mathbf{r}=\int_{C_1}\mathbf{F}\cdot d\mathbf{r}+\int_{C_2}\mathbf{F}\cdot d\mathbf{r}+\int_{C_3}\mathbf{F}\cdot d\mathbf{r}+\int_{C_4}\mathbf{F}\cdot d\mathbf{r}=2$$

(2) $$\oint_C\mathbf{F}\cdot d\mathbf{r}=\iint_S\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)dxdy=\int_0^1\int_0^1(2x+2y)dxdy=\int_0^1(x^2+2xy)\Big|_0^1dy$$

$$=\int_0^1(1+2y)dy=(y+y^2)\Big|_0^1=2$$

Ex. 15

Evaluate $\oint_C [(x^2-y^2)dx+(2y-x)dy]$, C consists the boundary of the region in the first qradrant that is bounded by the graphs of $y=x^2$ and $y=x^3$. [104中原機械甲7]



[解]$\mathbf{F}=(x^2-y^2)\mathbf{i}+(2y-x)\mathbf{j}\Rightarrow F_1=x^2-y^2,\ F_2=2y-x$

(1)在 $C_1: y=x^3, \mathbf{r}=x\mathbf{i}+y\mathbf{j}=x\mathbf{i}+x^3\mathbf{j}\Rightarrow d\mathbf{r}=(\mathbf{i}+3x^2\mathbf{j})dx$

$\mathbf{F}=(x^2-x^6)\mathbf{i}+(2x^3-x)\mathbf{j}$

$\int_{C_1}\mathbf{F}\cdot d\mathbf{r}=\int_0^1[(x^2-x^6)\mathbf{i}+(2x^3-x)\mathbf{j}]\cdot(\mathbf{i}+3x^2\mathbf{j})dx$

$=\int_0^1[(x^2-x^6)+3x^2(2x^3-x)dx=\int_0^1(-x^6+6x^5-3x^3+x^2)dx=\dfrac{37}{84}$

在 $C_2: y=x^2, \mathbf{r}=x\mathbf{i}+y\mathbf{j}=x\mathbf{i}+x^2\mathbf{j}\Rightarrow d\mathbf{r}=(\mathbf{i}+2x\mathbf{j})dx$

$\mathbf{F}=(x^2-x^4)\mathbf{i}+(2x^2-x)\mathbf{j}$

$\int_{C_2}\mathbf{F}\cdot d\mathbf{r}=\int_1^0[(x^2-x^4)\mathbf{i}+(2x^2-x)\mathbf{j}]\cdot(\mathbf{i}+2x\mathbf{j})dx$

$=\int_1^0[(x^2-x^4)+2x(2x^2-x)dx=\int_1^0(-x^4+4x^3-x^2)dx=-\dfrac{7}{15}$

$\int_C\mathbf{F}\cdot d\mathbf{r}=\int_{C_1}\mathbf{F}\cdot d\mathbf{r}+\int_{C_2}\mathbf{F}\cdot d\mathbf{r}=\dfrac{37}{84}-\dfrac{7}{15}=-\dfrac{11}{420}$

(2)由Green定理知

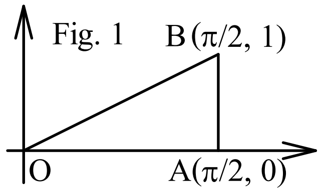
$\int_C\mathbf{F}\cdot d\mathbf{r}=\iint_S(\dfrac{\partial F_2}{\partial x}-\dfrac{\partial F_1}{\partial y})dxdy=\iint_S(-1+2y)]dxdy=\int_0^1\int_{x^3}^{x^2}(-1+2y)dydx$

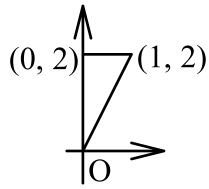
$=\int_0^1(-y+y^2)\Big|_{x^3}^{x^2}dx=\int_0^1(-x^2+x^3+x^4-x^6)dx=-\dfrac{11}{420}$

[Exercise]1. Evaluate the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ counterclockwise around the boundary C of the region R, where $\mathbf{F}=x\sin y\mathbf{i}-y\sin x\mathbf{j}$, R is the rectangle: $0\le x\le \pi,\ 0\le y\le \pi/2$.

2. Evaluate $\oint_C (y-\sin x)dx+\cos x dy$, where C is the triangle of the adjoining figure as shown in Fig. 1: (a)directly; (b) by using Green's theorem in the plane. [98成大機械]

3. 限用Green定理求$\oint_C 4x^2 y dx + 2y dy$之值，其中積分路徑C為以(0, 2), (0, 0), (1, 2)三點為頂點之三角形邊界(逆時針方向)，如圖。[91嘉大土木4]





1. [解](1)在C_1: $y=0,\ dy=0,\ \mathbf{F}=0 \Rightarrow \int_{C_1}\mathbf{F}\cdot d\mathbf{r}=0$;

在C_2: $x=\pi,\ dx=0,\ \mathbf{F}=\pi\sin y\mathbf{i} \Rightarrow \int_{C_2}\mathbf{F}\cdot d\mathbf{r}=0$;

在C_3: $y=\pi/2,\ dy=0,\ \mathbf{F}=x\mathbf{i}-\frac{\pi}{2}\sin x\mathbf{j} \Rightarrow \int_{C_3}\mathbf{F}\cdot d\mathbf{r}=\int_\pi^0 x dx=-\frac{\pi^2}{2}$;

在C_4: $x=0,\ dx=0,\ \mathbf{F}=0 \Rightarrow \int_{C_4}\mathbf{F}\cdot d\mathbf{r}=0$.

$$\int_C \mathbf{F}\cdot d\mathbf{r}=\int_{C_1}\mathbf{F}\cdot d\mathbf{r}+\int_{C_2}\mathbf{F}\cdot d\mathbf{r}+\int_{C_3}\mathbf{F}\cdot d\mathbf{r}+\int_{C_4}\mathbf{F}\cdot d\mathbf{r}=-\frac{\pi^2}{2}.$$

(2) $$\int_C \mathbf{F}\cdot d\mathbf{r}=\iint_S\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)dxdy=\int_0^{\pi/2}\int_0^{\pi}(-y\cos x - x\cos y)dxdy$$

$$=\int_0^{\pi/2}\left(-y\sin x-\frac{x^2}{2}\cos y\right)_0^\pi dy=\int_0^{\pi/2}-\frac{\pi^2}{2}\cos y dy=-\frac{\pi^2}{2}\sin y\Big|_0^{\pi/2}=-\frac{\pi^2}{2}.$$

2.[解](a)在$\overline{\mathrm{OA}}: y=0,\ dy=0$，積分為$\int_0^{\frac{\pi}{2}}(-\sin x)dx=\cos x\Big|_0^{\frac{\pi}{2}}=-1$

在$\overline{\mathrm{AB}}: x=\frac{\pi}{2},\ dx=0$，積分值為0

在$\overline{\mathrm{BO}}: y=\frac{2}{\pi}x$，積分為$\int_{\frac{\pi}{2}}^0\left(\frac{2}{\pi}x-\sin x\right)dx+\cos x\cdot\frac{2}{\pi}dx$

$$=\left(\frac{1}{\pi}x^2+\cos x+\frac{2}{\pi}\sin x\right)\Big|_{\frac{\pi}{2}}^0=-\frac{\pi}{4}+1-\frac{2}{\pi}$$

$$\oint_C(y-\sin xdx)+\cos xdy=-1+0-\frac{\pi}{4}+1-\frac{2}{\pi}=-\frac{\pi}{4}-\frac{2}{\pi}$$

(b)由Green定理知此積分為

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{2}{\pi}x}\left[\frac{\partial\cos x}{\partial x}-\frac{\partial}{\partial y}(y-\sin x)\right]dydx=\int_0^{\frac{\pi}{2}}\int_0^{\frac{2}{\pi}x}(-\sin x-1)dydx$$

$$=\int_0^{\frac{\pi}{2}}y(-\sin x-1)\Big|_0^{\frac{2}{\pi}x}dx=-\frac{2}{\pi}\int_0^{\frac{\pi}{2}}(x\sin x+x)dx$$

$$=-\frac{2}{\pi}\left(-x\cos x+\sin x+\frac{x^2}{2}\right)\Big|_0^{\frac{\pi}{2}}=-\frac{2}{\pi}\left(1+\frac{\pi^2}{8}\right)=-\frac{2}{\pi}-\frac{\pi}{4}$$

3.[解]$\mathbf{F} = 4x^2y\mathbf{i} + 2y\mathbf{j} \Rightarrow F_1 = 4x^2y, F_2 = 2y \Rightarrow \frac{\partial F_2}{\partial x} = 0, \frac{\partial F_1}{\partial y} = 4x^2$

$$\oint_C F_1dx + F_2dy = \iint_S (\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y})dxdy = \int_0^1\int_{2x}^2 (0 - 4x^2)dydx = \int_0^1 (-4x^2y)\Big|_{2x}^2 dx$$

$$= \int_0^1 (-8x^2 + 8x^3)dx = (-\frac{8}{3}x^3 + 2x^4)\Big|_0^1 = -\frac{2}{3}$$