Fourier Analysis

第一章 基本觀念

I. 週期函數(Periodic Function)

定義：對所有的實數 x，若存在正數 p，使得函數 $f(x)$ 滿足

$$f(x+p)=f(x) \tag{1.1}$$

$f(x)$ 稱為週期函數，其中 p 的最小值稱為 $f(x)$ 的週期。

Ex. 1

$\cos x$ and $\sin x$ have period of 2π.

Ex. 2

If $f(x)$ has a period of p, determine the period of $f(kx)$.

Solution: Since $f(x+p)=f(x)$, let $x=kt$, then

$$f(kt)=f(kt+p)=f(k(t+\frac{p}{k})).$$

Let $F(t)=f(kt)$, the above equation becomes

$$F(t)=F(t+\frac{p}{k}) \Rightarrow f(kx) \text{ has period of } \frac{p}{k}.$$

Ex. 3

If $f(x+p)=f(x)$, prove (1) $\int_p^{p+a} f(x)dx = \int_0^a f(x)dx$, and (2) $\int_{a-\frac{p}{2}}^{a+\frac{p}{2}} f(x)dx = \int_{-\frac{p}{2}}^{\frac{p}{2}} f(x)dx$.

Solution: $\int_\alpha^\beta f(x)dx = \int_\alpha^\beta f(x+p)dx$.

Let $x+p=t \Rightarrow x=t-p$, substituting into the right hand side, we have

$$\int_\alpha^\beta f(x)dx = \int_{\alpha+p}^{\beta+p} f(t)dt.$$

Noting that $\int_{\alpha+p}^{\beta+p} f(t)dt = \int_{\alpha+p}^{\beta+p} f(x)dx$, we get $\int_\alpha^\beta f(x)dx = \int_{\alpha+p}^{\beta+p} f(x)dx.$ $\qquad (a)$

(1) Let $\alpha = 0, \beta = a$, equation $(a) \Rightarrow \int_0^a f(x)dx = \int_p^{a+p} f(x)dx.$

(2) $\int_{a-\frac{p}{2}}^{a+\frac{p}{2}} f(x)dx = \int_{a-\frac{p}{2}}^{-\frac{p}{2}} f(x)dx + \int_{-\frac{p}{2}}^{a+\frac{p}{2}} f(x)dx,$

but $\int_{a-\frac{p}{2}}^{-\frac{p}{2}} f(x)dx = \int_{a+\frac{p}{2}}^{\frac{p}{2}} f(x)dx$, the above equation becomes

$$\int_{a-\frac{p}{2}}^{a+\frac{p}{2}} f(x)dx = \int_{a+\frac{p}{2}}^{\frac{p}{2}} f(x)dx + \int_{-\frac{p}{2}}^{a+\frac{p}{2}} f(x)dx = \int_{-\frac{p}{2}}^{\frac{p}{2}} f(x)dx.$$

II. 正交(Orthogonality)

定義：兩函數 $f(x)$及 $g(x)$對權值函數 $r(x)$在區間 $x\in[a, b]$的內積為

$$<f, g>_r=\int_a^b r(x)f(x)g(x)dx\,. \tag{1.2}$$

若 $<f, g>_r=0$，稱 $f(x)$及 $g(x)$對權值函數 $r(x)$在區間 $x\in[a, b]$為正交(orthogonal)。當 $r(x)=1$，(1.2)式可簡單地寫為

$$<f, g>=\int_a^b f(x)g(x)dx\,. \tag{1.3}$$

為了簡便，我們限制權值函數 $r(x)=1$。

定義：$f(x)$在區間 $x\in[a, b]$的範數為

$$||f(x)||=<f(x), f(x)>^{1/2}=\sqrt{\int_a^b f^2(x)dx}$$

若一組函數 $\phi_1(x), \phi_2(x),\cdots, \phi_n(x)$在區間 $x\in[a, b]$滿足

$$<\phi_i(x), \phi_j(x)>=\delta_{ij}\|\phi_i(x)\|^2, \tag{1.4}$$

吾人稱此組函數在區間 $x\in[a, b]$為正交函數(orthogonal function)，令 $u_i(x)=\dfrac{\phi_i(x)}{\|\phi_i(x)\|}$，則

$$<u_i(x), u_j(x)>=\delta_{ij}.$$

此組函數 $u_1(x), u_2(x),\cdots, u_n(x)$形成區間 $x\in[a, b]$的單位正交函數(orthonormal function)。

Ex. 4

Show that 1, cos$x$, sin$x$, cos$2x$, sin$2x$,$\cdots$, cosnx, sinnx are orthogonal on the interval $x \in [0, 2\pi]$.

Solution: $\int_0^{2\pi} \cos nx dx = \left.\frac{\sin nx}{n}\right|_0^{2\pi} = \begin{cases} 0, & n \neq 0 \\ 2\pi, & n = 0 \end{cases}$,

$$\int_0^{2\pi} \sin nx dx = -\left.\frac{\cos nx}{n}\right|_0^{2\pi} = 0,$$

$$\int_0^{2\pi} \cos mx \sin nx dx = \frac{1}{2}\int_0^{2\pi} [\sin(m+n)x + \sin(m-n)x] dx$$

$$= \frac{1}{2}\left[-\frac{\cos(m+n)x}{m+n} - \frac{\cos(m-n)x}{m-n}\right]_0^{2\pi} = 0,$$

$$\int_0^{2\pi} \cos mx \cos nx dx = \frac{1}{2}\int_0^{2\pi} [\cos(m+n)x + \cos(m-n)x] dx$$

$$= \frac{1}{2}\left[\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n}\right]_0^{2\pi} = \begin{cases} 0, & m \neq n \\ \pi, & m = n \neq 0 \\ 2\pi, & m = n = 0 \end{cases},$$

$$\int_0^{2\pi} \sin mx \sin nx dx = \frac{1}{2}\int_0^{2\pi} [\cos(m-n)x - \cos(m+n)x] dx$$

$$= \frac{1}{2}\left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n}\right]_0^{2\pi} = \begin{cases} 0, & m \neq n \\ \pi, & m = n \end{cases}.$$

Ex. 5

Form a set of orthogonal functions from $\{1, x, x^2, \cdots\}$ on the interval (0,1).

Solution: Let $\phi_1(x) = 1$, then

$$\|\phi_1(x)\| = \left(\int_0^1 1 \times 1 dx\right)^{1/2} = 1 \Rightarrow u_1(x) = \frac{\phi_1(x)}{\|\phi_1(x)\|} = 1,$$

$$\phi_2(x) = x - \langle x, u_1 \rangle u_1 = x - \left(\int_0^1 x \cdot 1 dx\right) \cdot 1 = x - \frac{1}{2},$$

$$\|\phi_2(x)\| = \left(\int_0^1 (x - \frac{1}{2})^2 dx\right)^{1/2} = \frac{1}{2\sqrt{3}} \Rightarrow u_2(x) = \frac{\phi_2(x)}{\|\phi_2(x)\|} = 2\sqrt{3}x - \sqrt{3},$$

$$\phi_3(x) = x^2 - \langle x^2, u_1 \rangle u_1 - \langle x^2, u_2 \rangle u_2 = x^2 - \left(\int_0^1 x^2 \cdot 1 dx\right) \cdot 1$$

$$-\left(\int_0^1 x^2 \cdot (2\sqrt{3}x - \sqrt{3}) dx\right)(2\sqrt{3}x - \sqrt{3}) = x^2 - x + \frac{1}{6}.$$

We obtain the orthogonal functions: $\{1, x - \frac{1}{2}, x^2 - x + \frac{1}{6}, \cdots\}$.

III. 偶函數與奇函數(Even and Odd Functions)

定義：對所有的 x，函數 $f(x)$稱為偶函數(even function)，若

$$f(x)=f(-x), \tag{1.5}$$

而函數 $f(x)$稱為奇函數(odd function)，若

$$f(x)=-f(-x). \tag{1.6}$$

即一偶函數的圖形會對稱於 y 軸，一奇函數的圖形會對稱於原點。任何函數皆可以表示成偶函數與奇函數的和，因為

$$f(x)=\frac{1}{2}[f(x)+f(-x)]+\frac{1}{2}[f(x)-f(-x)]=f_{\mathrm{e}}(x)+f_{\mathrm{o}}(x),$$

其中 $f_{\mathrm{e}}(x)=\frac{1}{2}[f(x)+f(-x)]$為偶函數，$f_{\mathrm{o}}(x)=\frac{1}{2}[f(x)-f(-x)]$為奇函數。

性質：1. 若 $f(x)$為偶函數，則$\int_{-c}^{c} f(x)dx=2\int_{0}^{c} f(x)dx$

2. 若 $f(x)$為奇函數，則$\int_{-c}^{c} f(x)dx=0$

3. $\begin{cases} 偶\pm偶,\ 偶\times\div偶,\ 奇\times\div奇 \Rightarrow 偶 \\ 奇\pm奇,\ 奇\times\div偶,\ 偶\times\div奇 \Rightarrow 奇 \end{cases}$

第二章 傅立葉級數(Fourier Series)

I. 傅立葉級數(Fourier Series)

定義：若有限多的點 $a = x_1 < x_2 < x_3 < \cdots < x_n = b$ 使得單值函數 $f(x)$在區間 $x_j < x < x_{j+1}$ 連續且單邊極限 $f(x_j^+)$及 $f(x_{j+1}^-)$存在，其中 $j = 1, 2, \cdots, n-1$， 則稱 $f(x)$在區間$[a, b]$為分段連續。

設 $f(x)$為週期 2π的週期函數，而 1, cos$x$, sin$x$, cos2x, sin2x,⋯ 在區間[−π, π]彼此正交，因此我們可以將分段連續函數 $f(x)$表為一級數：

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \tag{2.1}$$

其中

$$a_0 = \frac{2\int_{-\pi}^{\pi} f(x)dx}{\int_{-\pi}^{\pi} dx} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$a_n = \frac{\int_{-\pi}^{\pi} f(x)\cos nxdx}{\int_{-\pi}^{\pi} \cos^2 nxdx} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx, \qquad n = 1, 2, 3, \cdots, \tag{2.2}$$

$$b_n = \frac{\int_{-\pi}^{\pi} f(x)\sin nxdx}{\int_{-\pi}^{\pi} \sin^2 nxdx} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nxdx, \qquad n = 1, 2, 3, \cdots.$$

(2.2)式中的數稱為 $f(x)$的 Fourier 係數，三角級數(2.1)式稱為 $f(x)$的 Fourier 級數。

Ex. 1

There is a periodic square wave with analytically represented as $f(x)$ function
$f(x)=\begin{cases}-k, & \text{if } -\pi<x<0\\ k, & \text{if } 0<x<\pi\end{cases}$, and $f(x+2\pi)=f(x)$. Please find the Fourier coefficients a_n, b_n and their series function to present the $f(x)$ function. [106 元智機械 7]

[解]$f(x)=\dfrac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$,

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=0,$$

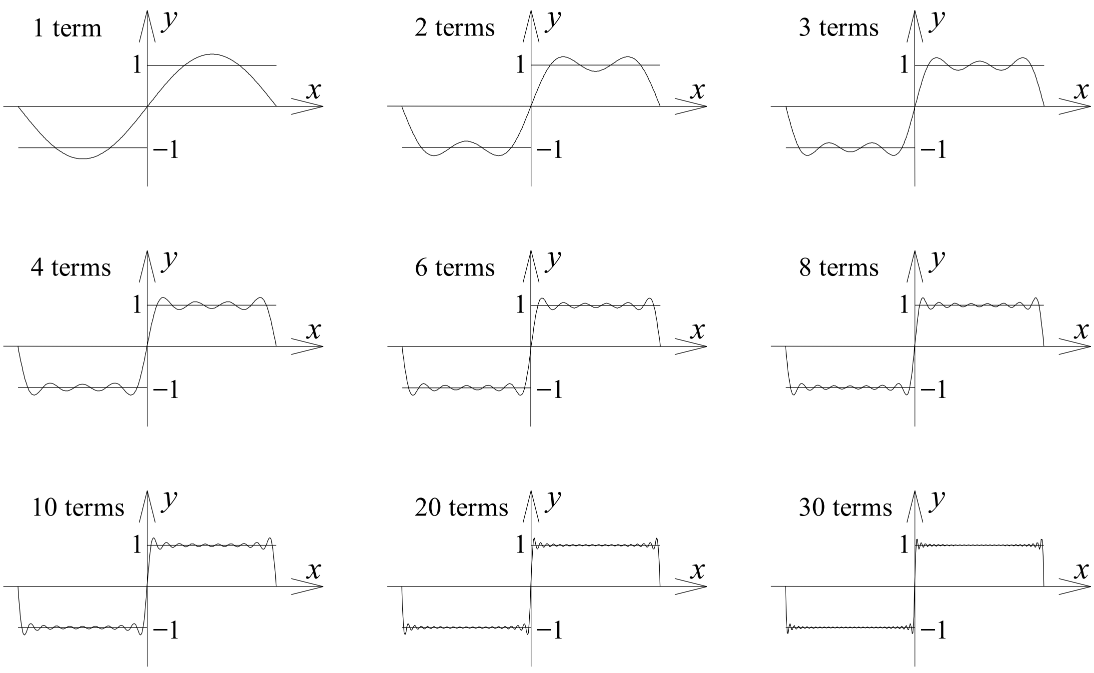
$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx=0,$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nxdx=\frac{2}{\pi}\int_{0}^{\pi}k\sin nxdx=\frac{2k}{n\pi}[1-(-1)^n]$$

$$=\begin{cases}0, & \text{for even } n\\ \dfrac{4k}{n\pi}, & \text{for odd } n\end{cases}=\frac{4k}{(2n-1)\pi}$$

$$f(x)=\frac{4k}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin(2n-1)x.$$

此 Fourier 級數的圖形如下



Ex. 2

Find the Fourier expansion of the function $f(x)=x^2$; $(-\pi<x<\pi)$. [98 高應大機械 7]

[解] $f(x)=\dfrac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$,

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}x^2dx=\frac{2\pi^2}{3},$$

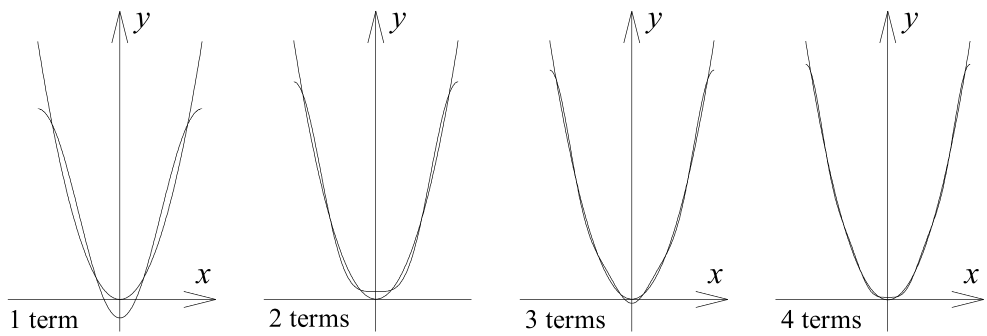
$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}x^2\cos nxdx=\frac{2}{\pi}\int_0^{\pi}x^2\cos nxdx=\frac{2}{n\pi}(x^2\sin nx\Big|_0^{\pi}-\int_0^{\pi}2x\sin nxdx)$$

$$=\frac{4}{n^2\pi}(x\cos nx\Big|_0^{\pi}-\int_0^{\pi}\cos nxdx)=\frac{4}{n^2\pi}[(-1)^n\pi]=(-1)^n\frac{4}{n^2},$$

$$b_n=0,$$

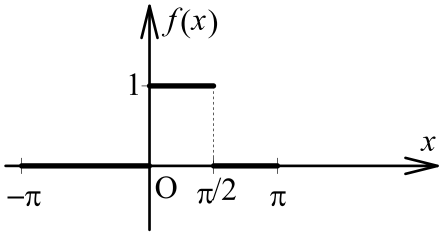
$$f(x)=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}(-1)^n\frac{\cos nx}{n^2},-\pi<x<\pi$$

$f(x)$的 Fourier 級數圖形如下：



Ex. 3

Find the Fourier series of the following function $f(x)$, which is assumed to have the period 2π. [104 高第一光電 7]



[解] $f(x)=\dfrac{a_0}{2}+\displaystyle\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$,

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{1}{\pi}\int_{0}^{\pi/2}1\cdot dx=\frac{1}{\pi}\cdot x\Big|_0^{\pi/2}=\frac{1}{\pi}\cdot\frac{\pi}{2}=\frac{1}{2}$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx=\frac{1}{\pi}\int_{0}^{\pi/2}1\cdot\cos nxdx=\frac{1}{n\pi}\cdot\sin nx\Big|_0^{\pi/2}=\frac{1}{n\pi}\cdot\sin\frac{n\pi}{2}$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nxdx=\frac{1}{\pi}\int_{0}^{\pi/2}1\cdot\sin nxdx=-\frac{1}{n\pi}\cdot\cos nx\Big|_0^{\pi/2}=-\frac{1}{n\pi}(\cos\frac{n\pi}{2}-1)$$

$$f(x)=\frac{1}{4}+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}[\sin\frac{n\pi}{2}\cos nx-(\cos\frac{n\pi}{2}-1)\sin nx]$$

[Exercises] 1. Find the Fourier series of the given function $f(x)$, which is assumed to have the period 2π, where $f(x)=0$ if $-\pi<x<0$, $f(x)=1$ if $0<x<\pi$. [100 嘉大資工 5]

2. If $f(t)=\sin\pi t$ for $t\in(-\pi,\pi]$ be a function of period 2π. Find the Fourier Series representation of $f(t)$. [98 台灣聯大 C]

3. Find the Fourier series of the function on the interval $[-\pi,\pi]$, $f(x)=-1$, $-\pi\le x\le 0$, and $f(x)=+1$, $0\le x\le\pi$. [103 海洋機械機電 6]

4. Find the Fourier series representation of the square wave which is given $f(x)=\begin{cases}3, & -\pi\le x<0\\ 5, & 0<x<\pi\end{cases}$ and $f(x+2\pi)=f(x)$. [86清大動機4]

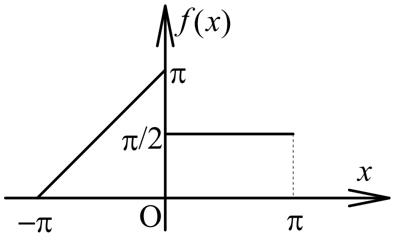
5. 一函數具有週期性：$f(x)=f(x+2\pi)$，其在$-\pi\le x\le\pi$之區間內定義為 $f(x)=\begin{cases}2, & -\pi\le x<0\\ 1, & 0\le x<\pi\end{cases}$. [101中原土木5]

6. $f(x)=x^2$, $0<x<2\pi$, $f(x)=f(x+2\pi)$. Find the Fourier series. [103 中央機械 6(a)]

7. Expand $f(x)=\begin{cases}0, & -\pi<x<0\\ \pi-x, & 0\le x<\pi\end{cases}$, in a Fourier series. [102 中原生醫 4、104 中原土木 3]

8. I. Write down the Fourier series expansion formula of a periodic function $f(t)$ with a period 2π. II. Determine the Fourier series representation of the periodic function $f(t)=e^t$ for $-\pi<t<\pi$ and $f(t+2\pi)=f(t)$. [104 中正機械 2(c)]

9. Find the Fourier series of the given function as shown, which is assumed to have the periodic 2π. [104 中原機械丙 6]



[Ans.]1. $f(x)=\frac{1}{2}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin(2n-1)x$ 2. $f(t)=\frac{2\sin\pi^2}{\pi}\sum_{n=1}^{\infty}\frac{n(-1)^n}{n^2-\pi^2}\sin nt$

3. $f(x)=\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin(2n-1)x,\quad -\pi<x<\pi$ 4. $f(x)=4+\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin(2n-1)x$

5. $f(x)=\frac{3}{2}-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin(2n-1)x$ 6. $f(x)=\frac{4\pi^2}{3}+4\sum_{n=1}^{\infty}(\frac{\cos nx}{n^2}-\frac{\pi\sin nx}{n})$

7. $f(x)=\frac{\pi}{4}+\frac{1}{\pi}\sum_{n=1}^{\infty}[\frac{1-(-1)^n}{n^2\pi}\cos nx+\frac{1}{n}\sin nx],\quad -\pi<x<\pi$

8. I. $f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nt+b_n\sin nt)$

II. $f(t)=\frac{\sinh\pi}{\pi}+\frac{2\sinh\pi}{\pi}\sum_{n=1}^{\infty}[(-1)^n(\cos nt+n\sin nt)]$

9. $f(x)=\frac{\pi}{2}+\frac{1}{\pi}\sum_{n=1}^{\infty}\left\{\frac{1-(-1)^n}{n^2}\cos nx-\frac{1}{n}\{\pi+\frac{\pi}{2}[(-1)^n-1]\}\sin nx\right\}$

II. 傳立葉餘弦及正弦級數(Fourier Cosine and Sine Series)

設 $f(x)$為在區間$[-\pi, \pi]$的偶函數，因為 cosnx 是偶函數，而 sinnx 是奇函數，函數 $f(x)\cos nx$ 是偶函數，$f(x)\sin nx$ 是奇函數，因此 $f(x)$的 Fourier 係數為

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx, \qquad n = 0, 1, 2, \cdots,$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = 0, \qquad n = 1, 2, 3, \cdots.$$

因此 $f(x)$的 Fourier 級數可寫成

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx, \tag{2.3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx, \qquad n = 0, 1, 2, 3, \cdots$$

稱為傳立葉餘弦級數(Fourier cosine series)。

同理，若 $f(x)$是奇函數，它的 Fourier 級數為

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx, \tag{2.4}$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx dx, \qquad n = 1, 2, 3, \cdots$$

稱為傳立葉正弦級數(Fourier sine series)。

Ex. 4

Find the Fourier series of the given function $f(x)$, which is assumed to have the period 2π.

$f(x)=\begin{cases} x+\pi, & \text{if } -\pi<x<0 \\ -x+\pi, & \text{if } 0<x<\pi \end{cases}$. [103中山材光6]

[解]$f(x)$為偶函數，其Fourier級數為

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx$$

$$a_0=\frac{2}{\pi}\int_0^{\pi}(-x+\pi)dx=\frac{2}{\pi}\cdot(-\frac{x^2}{2}+\pi x)\Big|_0^{\pi}=\frac{2}{\pi}\cdot(-\frac{\pi^2}{2}+\pi^2)=\pi$$

$$a_n=\frac{2}{\pi}\int_0^{\pi}(-x+\pi)\cos nxdx=\frac{2}{\pi}(\frac{1}{n})\left((-x+\pi)\sin nx\Big|_0^{\pi}+\int_0^{\pi}\sin nxdx\right)$$

$$=\frac{2}{n\pi}(0-\frac{\cos nx}{n}\Big|_0^{\pi})=-\frac{2}{n^2\pi}(\cos n\pi-1)=-\frac{2}{n^2\pi}[(-1)^n-1]=\frac{4}{(2n-1)^2\pi}$$

$$f(x)=\frac{\pi}{2}+\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos(2n-1)x$$

Ex. 5

Find the Fourier cosine series and Fourier sine series of $f(x)$, where $f(x)=\sin x$, $0<x<\pi$. [106暨南應光5]

[解](1)設 $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx$

$$a_0=\frac{2}{\pi}\int_0^{\pi}f(x)dx=\frac{2}{\pi}\int_0^{\pi}\sin xdx=\frac{2}{\pi}\cdot(-\cos x)\Big|_0^{\pi}=\frac{2}{\pi}\cdot(\cos\pi-1)=-\frac{4}{\pi}$$

$$a_n=\frac{2}{\pi}\int_0^{\pi}f(x)\cos nxdx=\frac{2}{\pi}\int_0^{\pi}\sin x\cos nxdx=\frac{1}{\pi}\int_0^{\pi}[\sin(1+n)x+\sin(1-n)x]dx$$

$$=-\frac{1}{\pi}[\frac{\cos(1+n)x}{1+n}+\frac{\cos(1-n)x}{1-n}]\Big|_0^{\pi}=-\frac{1}{\pi}[\frac{\cos(1+n)\pi-1}{1+n}+\frac{\cos(1-n)\pi-1}{1-n}]$$

$$=-\frac{1}{\pi}[\frac{(-1)^{1+n}-1}{1+n}+\frac{(-1)^{1-n}-1}{1-n}]=-\frac{1}{\pi}(\frac{-2}{1+2n}+\frac{-2}{1-2n})=\frac{2}{\pi}(\frac{1}{1+2n}+\frac{1}{1-2n})=\frac{4}{(1-4n^2)\pi}$$

$$f(x)=-\frac{2}{\pi}+\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{1-4n^2}\cos 2nx,\ 0<x<\pi$$

(2)設 $f(x)=\sum_{n=1}^{\infty}b_n\sin nx$, 其中$b_1=1,\ b_n=0,\ n\neq 1$

$$f(x)=\sin x,\ 0<x<\pi$$

[Exercises]1. Expand the given function in an appropriate cosine or sine Fourier series. $f(x)=|\sin x|$, $-\pi<x<\pi$. [103 逢甲電機 4]

2. (a)畫出週期函數$f(x)=\begin{cases}1, & 0<x<\pi \\ -1, & \pi<x<2\pi\end{cases}$且$f(x+2\pi)=f(x)$的圖形。(b)請問$f(x)$是奇函數、偶函數，還是非奇、非偶函數？(c)求$f(x)$的傅立葉級數。[103 逢甲光電 3]

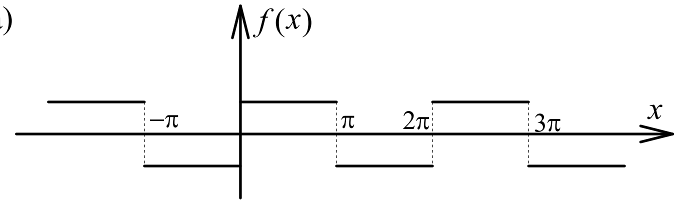
3. There is periodic square wave with analytic with represented as $f(x)$ function $f(x)$ $=\begin{cases}-k, & \text{when } -\pi<x<0 \\ k, & \text{when } 0<x<\pi\end{cases}$, and $f(x+2\pi)=f(x)$. Please find the Fourier coefficients a_n, b_n and their series functions to represent the $f(x)$ functions. [103 元智機械 7]

4. Find the Fourier series of the following function $f(x)=x+\pi$, if $-\pi<x<\pi$, and $f(x+2\pi)=f(x)$. [106 台大化工 1]

5. 試求函數$f(x)$的傅立葉餘弦級數(Fourier cosine series)。$f(x)=x$, $0<x<\pi$. [104 高第一環安 7]

[Ans.]1. $f(x)=\dfrac{2}{\pi}-\dfrac{8}{\pi}\sum_{n=1}^{\infty}\dfrac{n}{1-4n^2}\cos 2nx$

2.(a)



(b)奇函數　(c) $f(x)=\dfrac{4}{\pi}\sum_{n=1}^{\infty}\dfrac{1}{2n-1}\sin(2n-1)x$

3. $a_n=0$, $b_n=\dfrac{4k}{(2n-1)\pi}$, $f(x)=\dfrac{4k}{\pi}\sum_{n=1}^{\infty}\dfrac{1}{2n-1}\sin(2n-1)x$

4. $f(x)=\pi-2\sum_{n=1}^{\infty}\dfrac{(-1)^n}{n}\sin nx$　　5. $f(x)=\dfrac{\pi}{2}-\dfrac{4}{\pi}\sum_{n=1}^{\infty}\dfrac{1}{(2n-1)^2}\cos(2n-1)x$

III. 任意週期 $p=2\mathrm{L}$ 的函數

設 $f(x)$為週期 2L 的分段連續函數，令

$$\frac{x}{2\mathrm{L}}=\frac{t}{2\pi} \quad \Rightarrow \; x=\frac{\mathrm{L}}{\pi}t, \;\; t=\frac{\pi}{\mathrm{L}}x. \tag{2.5}$$

當$-\mathrm{L}<x<\mathrm{L}$ 時，$-\pi<t<\pi$，$f(x)=f(\frac{\mathrm{L}}{\pi}t)$；令 $F(t)=f(\frac{\mathrm{L}}{\pi}t)$，函數 $F(t)$的週期為 2π，它的 Fourier 級數為

$$F(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nt+b_n\sin nt), \tag{2.6}$$

其中

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}F(t)\cos nt\,dt, \qquad n=0,1,2,\cdots,$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}F(t)\sin nt\,dt, \qquad n=1,2,3,\cdots,$$

將(2.5)式代入，(2.6)式變為

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos\frac{n\pi x}{\mathrm{L}}+b_n\sin\frac{n\pi x}{\mathrm{L}}), \tag{2.7}$$

其中

$$a_n=\frac{1}{\mathrm{L}}\int_{-\mathrm{L}}^{\mathrm{L}}f(x)\cos\frac{n\pi x}{\mathrm{L}}dx, \qquad n=0,1,2,\cdots,$$

$$b_n=\frac{1}{\mathrm{L}}\int_{-\mathrm{L}}^{\mathrm{L}}f(x)\sin\frac{n\pi x}{\mathrm{L}}dx, \qquad n=1,2,3,\cdots.$$

Ex. 6

試求函數$f(x)=|\cos 2x|$的傅立葉級數。[106 中山環工 6]

[解]$\cos 2x$的週期為$\pi \Rightarrow |\cos 2x|$的週期為$\frac{\pi}{2}$且為偶函數

設 $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos 4nx$

$$a_0=\frac{8}{\pi}\int_0^{\frac{\pi}{4}} f(x)dx=\frac{8}{\pi}\int_0^{\frac{\pi}{4}}\cos 2x dx=\frac{8}{\pi}\cdot\left.\frac{\sin 2x}{2}\right|_0^{\frac{\pi}{4}}=\frac{4}{\pi}$$

$$a_n=\frac{8}{\pi}\int_0^{\frac{\pi}{4}} f(x)\cos 4nx dx=\frac{8}{\pi}\int_0^{\frac{\pi}{4}}\cos 2x\cos 4nx dx=\frac{4}{\pi}\int_0^{\frac{\pi}{4}}[\cos(4n+2)x+\cos(4n-2)x]dx$$

$$=\frac{4}{\pi}\cdot\left.[\frac{\sin(4n+2)x}{4n+2}+\frac{\sin(4n-2)x}{4n-2}]\right|_0^{\frac{\pi}{4}}=\frac{4}{\pi}\cdot[\frac{\sin(2n+1)\frac{\pi}{2}}{4n+2}+\frac{\sin(2n-1)\frac{\pi}{2}}{4n-2}]$$

$$=\frac{4}{\pi}\cdot[\frac{(-1)^n}{4n+2}+\frac{(-1)^{n-1}}{4n-2}]=\frac{4}{\pi}\cdot(-1)^{n-1}(\frac{-1}{4n+2}+\frac{1}{4n-2})=\frac{4}{\pi}\cdot(-1)^{n-1}\cdot\frac{1}{4n^2-1}$$

$$f(x)=\frac{2}{\pi}+\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{4n^2-1}\cos 4nx$$

Ex. 7

Find the Fourier series of the function $f(x)=\begin{cases} x, & 0\le x<1 \\ 0, & 1\le x<2 \end{cases}$. [91 成大造船 2]

[解]設 $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos n\pi x+b_n\sin n\pi x)$, 其中

$$a_0=\int_0^2 f(x)dx=\int_0^1 xdx+\int_1^2 0dx=\left.\frac{x^2}{2}\right|_0^1=\frac{1}{2}$$

$$a_n=\int_0^2 f(x)\cos n\pi x dx=\int_0^1 x\cos n\pi x dx=\frac{1}{n\pi}(x\sin n\pi x|_0^1-\int_0^1\sin n\pi x dx)$$

$$=\left.\frac{\cos n\pi x}{n^2\pi^2}\right|_0^1=\frac{\cos n\pi-1}{n^2\pi^2}=\frac{(-1)^n-1}{n^2\pi^2}$$

$$a_n=\int_0^2 f(x)\sin n\pi x dx=\int_0^1 x\sin n\pi x dx=-\frac{1}{n\pi}(x\cos n\pi x|_0^1-\int_0^1\cos n\pi x dx)$$

$$=-\frac{\cos n\pi}{n\pi}=-\frac{(-1)^n}{n\pi}$$

$$f(x)=\frac{1}{4}+\sum_{n=1}^{\infty}[\frac{(-1)^n-1}{n^2\pi^2}\cos n\pi x-\frac{(-1)^n}{n\pi}\sin n\pi x],\quad 0\le x<2$$

[Exercises]1. Expand $f(x)=x^2$ for $0<x<L$, (a) in a sine series, (b) in a cosine series, (c) in a Fourier series. [100 清大動機 7]

2. Find the Fourier series of $f(x)=\begin{cases}0, & -2<x<0\\ x, & 0\le x<1\\ 1, & 1\le x<2\end{cases}$. [94中央機械能源8]

3. Suppose a periodic function $f(t)$ with period is defined as $f(t)=\begin{cases}\frac{1}{k}, & 0\le t\le k\\ 0, & k\le t<2\end{cases}$, where k is a constant ($0<k<2$). Please expand $f(t)$ in a Fourier series. [100中原機械甲6]

4. Find the Fourier series of periodic function $f(x)$, $f(x)=-1(-1<x<0)$, $f(x)=1(0<x<1)$, $P=2L=2$. [102中原機械乙5]

5. Find the Fourier series of the function $f(x)=\begin{cases}-k, & -1<x<0\\ k, & 0<x<1\end{cases}$. [103中原機械丙8]

6. Find the Fourier series of the function, periodic square wave $f(t)=\begin{cases}0, & -2<t<-1\\ k, & -1<t<1\\ 0, & 1<t<2\end{cases}$. [105元智機械6]

7. If $f(t)=\begin{cases}2, & -2\le t<-1\\ 1, & -1\le t<1\\ 2, & 1\le t<2\end{cases}$, (a) find the Fourier series of $f(t)$, (b) find the value of the Fourier series, found in (a), converges to, when t is an integer, (c) find the steady state solution of the O.D.E.: $y''+25y=f(t)$, where $y''=d^2y/dt^2$. [98成大機械5]

[Ans.]1. (a) $-\frac{2}{\pi}\sum_{n=1}^{\infty}\{\frac{L^2(-1)^n}{n}-\frac{2L^2}{n^3\pi^2}[(-1)^n-1]\}\sin\frac{n\pi x}{L},\quad 0<x<L$

(b) $\frac{L^2}{3}+\frac{4L^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\frac{n\pi x}{L},\quad 0<x<L$

(c) $\frac{L^2}{3}+\sum_{n=1}^{\infty}(\frac{L^2}{n^2\pi^2}\cos\frac{2n\pi x}{L}-\frac{L^2}{n\pi}\sin\frac{2n\pi x}{L}),\quad 0<x<L$

2. $f(x)=\frac{3}{8}+\frac{1}{\pi^2}\sum_{n=1}^{\infty}\{(\frac{n\pi-2}{n^2}-\frac{\pi}{n}\sin\frac{n\pi}{2})\cos\frac{n\pi x}{2}+[\frac{2}{n^2}\sin\frac{n\pi}{2}-\frac{\pi}{n}(-1)^n]\sin\frac{n\pi x}{2}\},\ -2<x<2$

3. $f(t)=\frac{1}{2}+\frac{1}{k\pi}\sum_{n=1}^{\infty}[\frac{\sin kn\pi}{n}\cos n\pi t-\frac{\cos kn\pi-1}{n}\sin n\pi t]$

4. $f(x)=\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin(2n-1)\pi x$

5. $f(x)=\frac{4k}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin(2n-1)\pi x$

6. $f(t)=\frac{k}{2}+\frac{2k}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}\cos\frac{n\pi t}{2}$

7. (a) $\frac{3}{2}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}\cos\frac{(2n-1)\pi t}{2},\quad -2<t<2$

(b)$t=-2$：級數值 $\dfrac{f(-2^-)+f(-2^+)}{2}=\dfrac{2+2}{2}=2$

$t=-1$：級數值 $\dfrac{f(-1^-)+f(-1^+)}{2}=\dfrac{2+1}{2}=\dfrac{3}{2}$

$t=0$：級數值 $\dfrac{f(0^-)+f(0^+)}{2}=\dfrac{1+1}{2}=1$

$t=1$：級數值 $\dfrac{f(1^-)+f(1^+)}{2}=\dfrac{1+2}{2}=\dfrac{3}{2}, \qquad f(t+4)=f(t)$

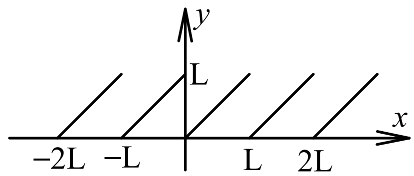
(c) $C_1\cos 5t + C_2\sin 5t + \dfrac{3}{50} + \dfrac{8}{\pi}\sum_{n=1}^{\infty}\dfrac{(-1)^n}{(2n-1)[100-(2n-1)^2\pi^2]}\cos\dfrac{(2n-1)\pi t}{2}$

IV. 全幅及半幅展開(Full Range and Half Range Expansion)

若$f(x)$只定義在某個區間，假設在 $0 \le x \le \mathrm{L}$，(1)我們可以將$f(x)$以週期 L 展開，這樣就是全幅展開(full range expansion)；(2)也可以先將$f(x)$從 $0 \le x \le \mathrm{L}$ 擴展成在$-\mathrm{L} \le x \le \mathrm{L}$ 的偶函數，再將$f(x)$展開成 Fourier 餘弦級數，此時，$f(x)$是一個週期 2L 的週期函數；(3)或先將$f(x)$從 $0 \le x \le \mathrm{L}$ 擴展成在$-\mathrm{L} \le x \le \mathrm{L}$ 的奇函數，再將$f(x)$展開成 Fourier 正弦級數，此時，$f(x)$也是一個週期 2L 的週期函數。

Ex. 8

Represent $f(x)$ by a Fourier series, $f(x) = x,\ 0 < x < \mathrm{L}$.



Solution: (1) Full range expansion

$$\text{Let } f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{2n\pi x}{\mathrm{L}} + b_n \sin\frac{2n\pi x}{\mathrm{L}}\right)$$

$$a_0 = \frac{2}{\mathrm{L}}\int_0^{\mathrm{L}} f(x)dx = \frac{2}{\mathrm{L}}\int_0^{\mathrm{L}} x dx = \mathrm{L}$$

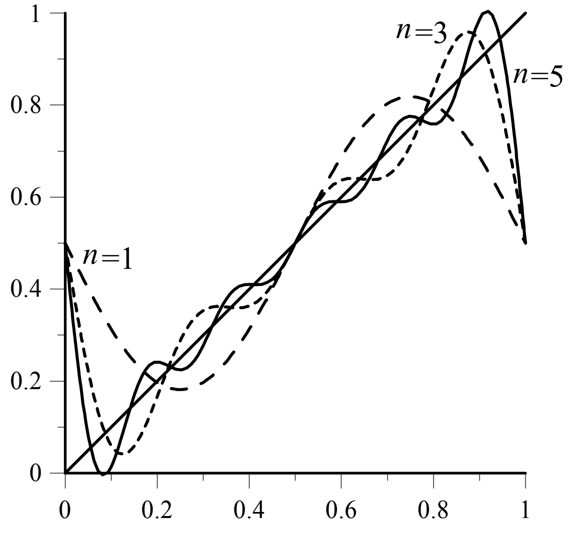
$$a_n = \frac{2}{\mathrm{L}}\int_0^{\mathrm{L}} f(x)\cos\frac{2n\pi x}{\mathrm{L}}dx = \frac{2}{\mathrm{L}}\int_0^{\mathrm{L}} x\cos\frac{2n\pi x}{\mathrm{L}}dx$$

$$= \frac{1}{n\pi}\left(x\sin\frac{2n\pi x}{\mathrm{L}}\Big|_0^{\mathrm{L}} - \int_0^{\mathrm{L}}\sin\frac{2n\pi x}{\mathrm{L}}dx\right) = \frac{\mathrm{L}}{2n^2\pi^2}\cos\frac{2n\pi x}{\mathrm{L}}\Big|_0^{\mathrm{L}} = 0$$

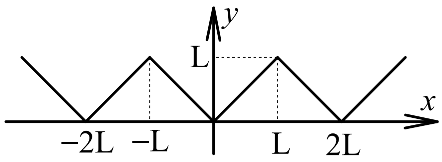
$$b_n = \frac{2}{\mathrm{L}}\int_0^{\mathrm{L}} f(x)\sin\frac{2n\pi x}{\mathrm{L}}dx = \frac{2}{\mathrm{L}}\int_0^{\mathrm{L}} x\sin\frac{2n\pi x}{\mathrm{L}}dx$$

$$= -\frac{1}{n\pi}\left(x\cos\frac{2n\pi x}{\mathrm{L}}\Big|_0^{\mathrm{L}} - \int_0^{\mathrm{L}}\cos\frac{2n\pi x}{\mathrm{L}}dx\right) = -\frac{\mathrm{L}}{n\pi} + \frac{\mathrm{L}}{2n^2\pi^2}\sin\frac{2n\pi x}{\mathrm{L}}\Big|_0^{\mathrm{L}} = -\frac{\mathrm{L}}{n\pi}$$

$$f(x) = \frac{\mathrm{L}}{2} - \frac{\mathrm{L}}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{2n\pi x}{\mathrm{L}}$$



(2)Fourier cosine series



Let $f(x)=\dfrac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\dfrac{n\pi x}{L}$

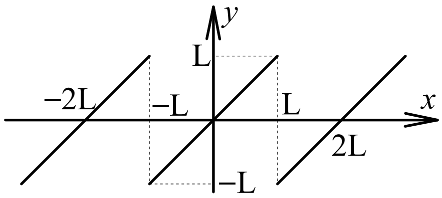
$$a_0=\frac{2}{L}\int_0^L x\,dx=L$$

$$a_n=\frac{2}{L}\int_0^L x\cos\frac{n\pi x}{L}dx=\frac{2}{n\pi}\left(x\sin\frac{n\pi x}{L}\Big|_0^L-\int_0^L\sin\frac{n\pi x}{L}dx\right)$$

$$=\frac{2L}{n^2\pi^2}\cos\frac{n\pi x}{L}\Big|_0^L=\frac{2L}{n^2\pi^2}[(-1)^n-1]=-\frac{4L}{(2n-1)^2\pi^2}$$

$$f(x)=\frac{L}{2}-\frac{4L}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\frac{(2n-1)\pi x}{L}$$

(3)Fourier sine series

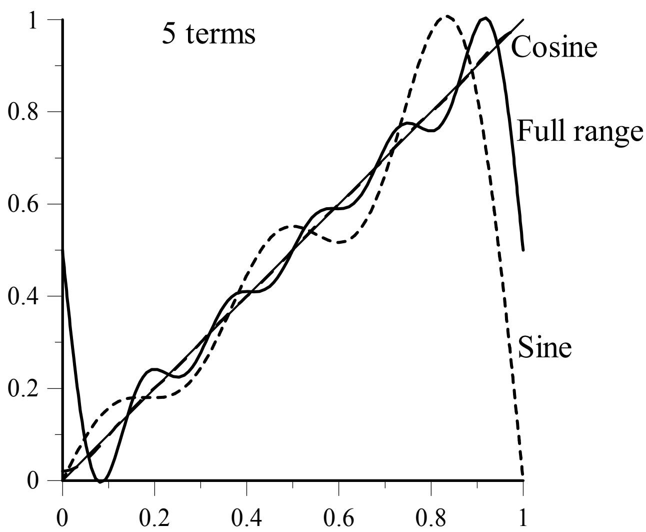


Let $f(x)=\sum_{n=1}^{\infty}b_n\sin\dfrac{n\pi x}{L}$

$$b_n=\frac{2}{L}\int_0^L x\sin\frac{n\pi x}{L}dx=-\frac{2}{n\pi}\left(x\cos\frac{n\pi x}{L}\Big|_0^L-\int_0^L\cos\frac{n\pi x}{L}dx\right)$$

$$=-\frac{2L}{n\pi}(-1)^n+\frac{2L}{n^2\pi^2}\sin\frac{n\pi x}{L}\Big|_0^L=\frac{(-1)^{n-1}2L}{n\pi}$$

$$f(x)=\frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sin\frac{n\pi x}{L}$$



Ex. 9

若$f(x)$的傅立葉級數為 $a_0+\sum_{n=1}^{\infty} a_n\cos(n\pi x)+b_n\sin(n\pi x)$ ，試求 $f(x)=\begin{cases} x+1, & -1<x<0 \\ x-1, & 0\le x<1 \end{cases}$ ，

且 $f(x)=f(x+2)$之(a)a_0及 a_n，(b)b_n。[102 虎尾電機 2]

[解]$f(x)$為奇函數 $\Rightarrow a_0=0,\ a_n=0,\ f(x)=\sum_{n=1}^{\infty} b_n \sin n\pi x$

$$b_n=2\int_0^1 f(x)\sin n\pi x dx=2\int_0^1 (x-1)\sin n\pi x dx=2(\int_0^1 x\sin n\pi x dx-\int_0^1 \sin n\pi x dx)$$

$$=-\frac{2}{n\pi}[(x\cos n\pi x\Big|_0^1-\int_0^1 \cos n\pi x dx)-\cos n\pi x\Big|_0^1]=-\frac{2}{n\pi}[\cos n\pi-(\cos n\pi-1)]=-\frac{2}{n\pi}$$

[Exercises]1. 試求函數$f(x)$的傅立葉餘弦級數(Fourier cosine series)。$f(x)=x,\ 0<x<\pi$ [104 高第一環安衛 7]

2. Given $f(x)=L-x,\ 0<x<L$, represent $f(x)$ by a Fourier sine series.
Hint: $f(x)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$, where $b_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx,\ n=1,2,\cdots$ [102中原機械丙6]

3. Find the Fourier series of the function $f(x)=x+5,\ -1<x<1,\ f(x)=f(x+2)$. [105南大綠能7]

4. 對一函數$f(x)$，$-L\le x\le L$，可用下列的傅立葉級數展開：
$f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty} a_n\cos(\frac{n\pi x}{L})+b_n\sin(\frac{n\pi x}{L})$；若$f(x)=\begin{cases} -2, & -\pi\le x\le 0 \\ 2, & 0<x\le\pi \end{cases}$，$(L=\pi)$，求$a_0$，$a_n$及$b_n$。[102虎尾電子5]

5. 已知$f(x)=1$，$0<x<\pi$，請完成下列工作，
(a)寫出餘弦半幅擴張$\frac{1}{2}a_0+\sum_{n=1}^{\infty} a_n\cos nx$ ，其中$a_0=\frac{2}{\pi}\int_0^{\pi} f(x)dx$ ，
$a_n=\frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx,\ n=1,2,3,\cdots$；(b)寫出正弦半幅擴張$\sum_{n=1}^{\infty} b_n\sin nx$ ，其中$b_n=\frac{2}{\pi}\int_0^{\pi} f(x)\sin nx dx,\ n=1,2,3,\cdots$。[104高海輪機七]

[Ans.]1. $f(x)=\frac{\pi}{2}-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2},\ 0<x<\pi$ 2. $f(x)=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{L},\ 0<x<L$

3. $f(x)=5-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin n\pi x$ 4. $a_0=a_n=0,\ b_n=\frac{8}{(2n-1)\pi}$

5. (a)$f(x)=1$ (b)$f(x)=\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{2n-1}$

V. 複數型 Fourier 級數(Complex Fourier Series)

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\frac{e^{inx}+e^{-inx}}{2}+b_n\frac{e^{inx}-e^{-inx}}{2i})$$

$$=\frac{a_0}{2}+\sum_{n=1}^{\infty}[(\frac{a_n-ib_n}{2})e^{inx}+(\frac{a_n+ib_n}{2})e^{-inx}]=c_0+\sum_{n=1}^{\infty}(c_ne^{inx}+c_{-n}e^{-inx}]$$

其中

$$c_0=\frac{a_0}{2}=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)dx$$

$$c_n=\frac{a_n-ib_n}{2}=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)(\cos nx-i\sin nx)dx=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)e^{-inx}dx$$

$$c_{-n}=\frac{a_n+ib_n}{2}=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)(\cos nx+i\sin nx)dx=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)e^{inx}dx$$

因此

$$f(x)=\sum_{n=-\infty}^{\infty}c_ne^{inx}\qquad -\pi<x<\pi$$

其中

$$c_n=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)e^{-inx}dx$$

對於週期 2L 的複數型 Fourier 級數為

$$f(x)=\sum_{n=-\infty}^{\infty}c_ne^{i\frac{n\pi x}{\mathrm{L}}}\qquad -\mathrm{L}<x<\mathrm{L}\tag{2.8}$$

其中

$$c_n=\frac{1}{2\mathrm{L}}\int_{-\mathrm{L}}^{\mathrm{L}}f(x)e^{-i\frac{n\pi x}{\mathrm{L}}}dx$$

Ex. 10

Expand $f(x)=\begin{cases}-1, & -\pi<x<0\\ 1, & 0\le x<\pi\end{cases}$ in a complex Fourier series. [93 中央機械 8(b)]

[解]令 $f(x)=\sum_{n=-\infty}^{\infty}c_n e^{inx},\quad -\pi<x<\pi$

$$c_n=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)e^{-inx}dx=\frac{1}{2\pi}[\int_{-\pi}^{0}-1\cdot e^{-inx}dx+\int_{0}^{\pi}1\cdot e^{-inx}dx]=\frac{1}{2\pi}(\frac{e^{-inx}}{in}\bigg|_{-\pi}^{0}+\frac{e^{-inx}}{-in}\bigg|_{0}^{\pi})$$

$$=\frac{1}{2\pi}(\frac{1-e^{in\pi}}{in}+\frac{e^{-in\pi}-1}{-in})=\frac{1}{2\pi}(\frac{2}{in}-\frac{e^{in\pi}}{in}-\frac{e^{-in\pi}}{in})$$

$$=\frac{1}{2\pi}[\frac{2}{in}-\frac{\cos n\pi+i\sin n\pi}{in}-\frac{\cos(-n\pi)+i\sin(-n\pi)}{in}]$$

$$=\frac{1}{2\pi}[\frac{2}{in}-\frac{(-1)^n}{in}-\frac{(-1)^n}{in}]=\frac{1}{\pi}[\frac{1-(-1)^n}{in}]=\frac{2}{i\pi(2n-1)}$$

$$\therefore e^x=\frac{2}{i\pi}\sum_{n=-\infty}^{\infty}\frac{1}{2n-1}e^{i(2n-1)x},\quad -\pi<x<\pi$$

[Exercises]1. Find the complex Fourier series of the $f(x)$ on the given interval. [104 清大生醫丙 8]

$$f(x)=\begin{cases}0, & -\frac{1}{2}<x<-\frac{1}{4}\\ 1, & -\frac{1}{4}<x<\frac{1}{4}\\ 0, & \frac{1}{4}<x<\frac{1}{2}\end{cases}$$

2. 週期函數$x(t)=2\sin 3\pi t\cos\pi t+4\cos^2\pi t$，試寫出$x(t)$的複指數傅立葉級數(Complex Fourier series)。[102虎尾飛機乙4]

[Ans.]1. $f(x)=\frac{2}{i\pi}\sum_{n=-\infty}^{\infty}\frac{1}{n}\sinh\frac{n\pi}{2}e^{i2n\pi x},\quad -\frac{1}{2}<x<\frac{1}{2}$ 2. $f(x)=\pi+i\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{1}{n}e^{inx},\quad 0<x<2\pi$

3.

VI. Fourier Series 的收斂性(Convergence of Fourier Series)

1. 貝索不等式及 Parseval 等式(Bessel's inequality and Parseval's equality)

設 $f(x)$為週期 2π的分段連續函數，很顯然

$$\int_{-\pi}^{\pi}[f(x)-s_k(x)]^2\,dx \ge 0$$

其中

$$s_k(x)=\frac{a_0}{2}+\sum_{n=1}^{k}(a_n\cos nx+b_n\sin nx)$$

將其展開

$$\int_{-\pi}^{\pi}[f(x)-s_k(x)]^2\,dx=\int_{-\pi}^{\pi}[f(x)]^2\,dx-2\int_{-\pi}^{\pi}f(x)s_k(x)dx+\int_{-\pi}^{\pi}[s_k(x)]^2\,dx$$

然而，由 Fourier 係數的定義及正交的關係知

$$\int_{-\pi}^{\pi}f(x)s_k(x)dx=\int_{-\pi}^{\pi}f(x)[\frac{a_0}{2}+\sum_{n=1}^{k}(a_n\cos nx+b_n\sin nx)]dx=\frac{\pi a_0^2}{2}+\pi\sum_{n=1}^{k}(a_n^2+b_n^2)$$

及

$$\int_{-\pi}^{\pi}[s_k(x)]^2\,dx=\int_{-\pi}^{\pi}\{[\frac{a_0}{2}+\sum_{n=1}^{k}(a_n\cos nx+b_n\sin nx)][\frac{a_0}{2}+\sum_{n=1}^{k}(a_n\cos nx+b_n\sin nx)]\}dx$$
$$=\frac{\pi a_0^2}{2}+\pi\sum_{n=1}^{k}(a_n^2+b_n^2)$$

因此

$$\int_{-\pi}^{\pi}[f(x)-s_k(x)]^2\,dx=\int_{-\pi}^{\pi}[f(x)]^2\,dx-[\frac{\pi a_0^2}{2}+\pi\sum_{n=1}^{k}(a_n^2+b_n^2)]\ge 0$$

對所有 k 值，得到

$$\frac{a_0^2}{2}+\sum_{n=1}^{k}(a_n^2+b_n^2)\le\frac{1}{\pi}\int_{-\pi}^{\pi}[f(x)]^2\,dx$$

因為上式右手邊與 k 值無關，得到

$$\frac{a_0^2}{2}+\sum_{n=1}^{\infty}(a_n^2+b_n^2)\le\frac{1}{\pi}\int_{-\pi}^{\pi}[f(x)]^2\,dx \qquad (2.9)$$

此式稱為貝索不等式(Bessel's inequality)。

(2.9)式的左手邊非遞減且有界限，因此，級數

$$\frac{a_0^2}{2}+\sum_{n=1}^{\infty}(a_n^2+b_n^2) \tag{2.10}$$

收斂，得(2.10)式收斂的必要條件為

$$\lim_{n\to\infty} a_n = 0, \quad \lim_{n\to\infty} b_n = 0$$

當

$$\lim_{k\to\infty}\int_{-\pi}^{\pi}\left\{f(x)-\left[\frac{a_0}{2}+\sum_{n=1}^{k}(a_n\cos nx+b_n\sin nx)\right]\right\}^2 dx=0$$

稱 Fourier 級數收斂至 $f(x)$的均值，若 Fourier 級數收斂至 $f(x)$的均值，則

$$\frac{a_0^2}{2}+\sum_{n=1}^{\infty}(a_n^2+b_n^2)=\frac{1}{\pi}\int_{-\pi}^{\pi}[f(x)]^2\,dx \tag{2.11}$$

此式稱為 Parseval 等式。

2. Fourier 定理(Fourier theorem)

Fourier 級數收斂至對應函數的條件之定理稱為 Fourier 定理，設 $f(x)$在區間$(-\pi, \pi)$分段連續且週期為 2π的週期函數，它的 Fourier 級數

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)\text{收斂至}\frac{f(x^-)+f(x^+)}{2}.$$

若 $f(x)$只定義在區間$(-\pi, \pi)$為分段連續，此定理仍適用於先前所講將 $f(x)$作週期的延伸，亦即，在$-\pi\le x\le\pi$的內部，$f(x)$的 Fourier 級數收斂至均方值$\frac{f(x^-)+f(x^+)}{2}$，在兩端點 $x=\pm\pi$，收斂至$\frac{f(-\pi^+)+f(\pi^-)}{2}$。

Ex. 11

(a)Find the Fourier series of the function $f(x)$, where $f(x)=x^2$，$-\pi \le x \le \pi$. (b)Use the results in (a) to prove $\frac{\pi^2}{6}=\sum_{n=1}^{\infty}\frac{1}{n^2}=1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots$. (c)Use the results in (a) to calculate $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}=?$ [98 彰師大機電 4]

[解](1)令 $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$，因 $f(x)$為偶函數$b_n=0$

$$a_0=\frac{2}{\pi}\int_0^{\pi}x^2dx=\frac{2\pi^2}{3}$$

$$a_n=\frac{2}{\pi}\int_0^{\pi}x^2\cos nxdx=\frac{2}{n\pi}(x^2\sin nx\Big|_0^{\pi}-2\int_0^{\pi}x\sin nxdx)$$

$$=\frac{4}{n^2\pi}(x\cos nx\Big|_0^{\pi}-\int_0^{\pi}\cos nxdx)=(-1)^n\frac{4}{n^2}$$

$$x^2=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$$

(2)$x=\pi$代入上式得

$$\frac{f(\pi^-)+f(\pi^+)}{2}=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}(-1)^n\Rightarrow\pi^2=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{1}{n^2}\Rightarrow\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$$

Ex. 12

Show that the Fourier series of $f(x)=x$，$-\pi<x<\pi$ leads to $\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots$. [97 中央機械能源光機電生醫 8]

[解]$f(x)$為奇函數$\Rightarrow$令 $f(x)=\sum_{n=1}^{\infty}b_n\sin nx$

$$b_n=\frac{2}{\pi}\int_0^{\pi}x\sin nxdx=-\frac{2}{n\pi}(x\cos nx\Big|_0^{\pi}-\int_0^{\pi}\cos nxdx)=-\frac{2}{n\pi}(\pi\cos n\pi)=\frac{2(-1)^{n-1}}{n}$$

$$f(x)=2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sin nx$$

$x=\frac{\pi}{2}$代入上式得

$$\frac{f(\frac{\pi}{2}^-)+f(\frac{\pi}{2}^+)}{2}=2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sin\frac{n\pi}{2}\Rightarrow\frac{\pi}{2}=2(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots)$$

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots$$

[Exercises]

1. 令$f(x)=\dfrac{x^2}{2}$，$-\pi\le x\le\pi$，試以傅立葉級數展開並以此求級數$\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$。[97 虎尾機械 4]

2. Use the Fourier series of the function $f(x)=\begin{cases}0, & \text{if } 0<x<\pi \\ 1, & \text{if } \pi<x<2\pi\end{cases}$, find the sum $1-\dfrac{1}{3}+\dfrac{1}{5}-\dfrac{1}{7}+\cdots$. [101 宜蘭電機 5]

3. $f(x)=x^2$ is defined within $0<x<2\pi$, and $f(x)$ has a period 2π, then (a)find the Fourier series of $f(x)$, (b)evaluate the sum of the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ from result of (a), (c)evaluate the sum of the series $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$ from result of (a). [99 交大機械甲 6]

4. If $r(x)=x^2$, $0<x<2\pi$, $r(x)=r(x+2\pi)$. (a)Find $r(x)$ in the Fourier series. (b)Evaluate $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$. (c)Evaluate $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$. (d)Evaluate $\displaystyle\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$. [104 中央機械能源光機電 2]

5. (a)Find the Fourier series of periodic function $f(x)=\begin{cases}k, & \text{if } -\pi/2<x<\pi/2 \\ 0, & \text{if } \pi/2<x<3\pi/2\end{cases}$. (b)Show $1-\dfrac{1}{3}+\dfrac{1}{5}-\dfrac{1}{7}+\cdots=\dfrac{\pi}{4}$. [91 清大動機 5]

6. (a)Expand $f(x)=x+\pi$, $-\pi<x<\pi$ in a Fourier series. (b)Use the result of (a) to find $1-\dfrac{1}{3}+\dfrac{1}{5}-\dfrac{1}{7}+\cdots$. [94 中央機械 7]

7. 已知週期函數$f(t)=\begin{cases}0, & -\pi<t<-\dfrac{\pi}{2} \\ \pi, & -\dfrac{\pi}{2}<t<\dfrac{\pi}{2} \\ 0, & \dfrac{\pi}{2}<t<\pi\end{cases}$，$f(t)=f(t+2\pi)$，試求其傅立葉級數，並利用此結果證明等式$1-\dfrac{1}{3}+\dfrac{1}{5}-\dfrac{1}{7}+\cdots=\dfrac{\pi}{4}$。[104 屏科大車輛 7]

[Ans.] 1. $f(x)=\dfrac{\pi^2}{6}+2\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$, $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$

2. $f(x)=\dfrac{1}{2}-\dfrac{2}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin(2n-1)x$, $\dfrac{1}{1}-\dfrac{1}{3}+\dfrac{1}{5}-\dfrac{1}{7}+\cdots=\dfrac{\pi}{4}$

3. (a) $f(x)=\dfrac{4\pi^2}{3}+4\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\cos nx-\frac{\pi}{n}\sin nx\right)$ (b) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$ (c) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}=\frac{\pi^2}{12}$

4. (a) $f(x)=\dfrac{4\pi^2}{3}+4\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\cos nx-\frac{\pi}{n}\sin nx\right)$ (b) $\dfrac{\pi^2}{6}$ (c) $\dfrac{\pi^2}{12}$ (d) $\dfrac{\pi^2}{8}$

5. (a) $f(x)=\frac{k}{2}+\frac{2k}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}\cos(2n-1)x$ (b)略

6. (a) $f(x)=\pi-2\sum_{n=1}^{\infty}\frac{(-1)^{n}}{n}\sin nx$ (b) $\frac{\pi}{4}$

7. $f(t)=\frac{\pi}{2}+2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}\cos nt$

VII. Fourier 級數的微分與積分

1. 微分定理

設 (x)在區間$[-\pi, \pi]$為連續函數，且 $f(-\pi)=f(\pi)$，又 $f'(x)$在該區間為分段平滑，則 $f'(x)$ 的 Fourier 級數可由 $f(x)$的 Fourier 級數一項一項微分而得，且微分後的級數收斂至 $f'(x)$。

2. 積分定理

設 $f(x)$在區間$[-\pi, \pi]$分段連續且為週期 2π的函數，則無論 $f(x)$的 Fourier 級數收斂與否，皆可在任何上下限逐項積分。

已知 $f(x)=|\sin x|$ 的 Fourier 級數為

$$\sin x=\frac{2}{\pi}+\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nx}{1-4n^2}, \qquad 0<x<\pi$$

而 $f(x)=|\sin x|$在區間$[-\pi, \pi]$連續且 $f(-\pi)=f(\pi)$，因此，逐項微分得

$$\cos x=-\frac{8}{\pi}\sum_{n=1}^{\infty}\frac{n\sin 2nx}{1-4n^2}, \qquad 0<x<\pi$$

Ex. 13

已知 $f(x)=x,\ 0<x<2$ 表為傳立葉正弦級數為 $x=\sum_{n=1}^{\infty}\frac{-4}{n\pi}\cos(n\pi)\sin(\frac{n\pi x}{2})$，試利用積分求 $F(x)=x^2,\ 0<x<2$ 之傳立葉級數。[105 屏科大車輛 8]

[解]將 $f(x)$的 Fourier 級數積分得

$$\frac{x^2}{2}=\sum_{n=1}^{\infty}\frac{-4}{n\pi}\cos(n\pi)[-\frac{2}{n\pi}\cos(\frac{n\pi x}{2})]+k=\frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos(n\pi)}{n^2}\cos(\frac{n\pi x}{2})+k$$

$$x^2=\frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos(n\pi)}{n^2}\cos(\frac{n\pi x}{2})+C$$

$$x=0\text{代入}\Rightarrow 0=\frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos(n\pi)}{n^2}+C\cdots\cdots\cdots\text{(i)}$$

$$\text{而}\sum_{n=1}^{\infty}\frac{\cos(n\pi)}{n^2}=-\frac{1}{1^2}+\frac{1}{2^2}-\frac{1}{3^2}+\cdots=-\frac{\pi^2}{12}$$

$$\text{(i)}\Rightarrow 0=\frac{16}{\pi^2}\cdot(-\frac{\pi^2}{12})+C\Rightarrow C=\frac{4}{3}$$

$$\therefore x^2=\frac{4}{3}+\frac{16}{\pi^2}\sum_{n=1}^{\infty}\cos(n\pi)\cos(\frac{n\pi x}{2})$$

第三章 傅立葉積分

I. 傅立葉積分(Fourier Integral)

我們已描述過週期函數的 Fourier 級數，然而，非週期函數不能以 Fourier 級數表示，在許多問題上，仍渴望將函數像 Fourier 級數般發展成積分的表示式。$f(x)$在區間[−L, L]的 Fourier 級數為

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{\mathrm{L}}x+b_n\sin\frac{n\pi}{\mathrm{L}}x\right) \tag{3.1}$$

其中

$$a_n=\frac{1}{\mathrm{L}}\int_{-\mathrm{L}}^{\mathrm{L}}f(u)\cos\frac{n\pi u}{\mathrm{L}}du \qquad n=0,1,2,3,\cdots$$

$$b_n=\frac{1}{\mathrm{L}}\int_{-\mathrm{L}}^{\mathrm{L}}f(u)\sin\frac{n\pi u}{\mathrm{L}}du \qquad n=1,2,3,\cdots$$

代入(3.1)式，得

$$f(x)=\frac{1}{2\mathrm{L}}\int_{-\mathrm{L}}^{\mathrm{L}}f(u)du+\sum_{n=1}^{\infty}\frac{1}{\mathrm{L}}\int_{-\mathrm{L}}^{\mathrm{L}}f(u)\left(\cos\frac{n\pi u}{\mathrm{L}}\cos\frac{n\pi x}{\mathrm{L}}+\sin\frac{n\pi u}{\mathrm{L}}\sin\frac{n\pi x}{\mathrm{L}}\right)du$$

$$f(x)=\frac{1}{2\mathrm{L}}\int_{-\mathrm{L}}^{\mathrm{L}}f(u)du+\frac{1}{\mathrm{L}}\sum_{n=1}^{\infty}\int_{-\mathrm{L}}^{\mathrm{L}}f(u)\cos\frac{n\pi(u-x)}{\mathrm{L}}du \tag{3.2}$$

假設$f(x)$是絕對可積分，即$\int_{\infty}^{\infty}|f(u)|du$收斂，則

$$\frac{a_0}{2}=\frac{1}{2\mathrm{L}}\int_{-\mathrm{L}}^{\mathrm{L}}f(u)du\le\frac{1}{2\mathrm{L}}\left|\int_{-\mathrm{L}}^{\mathrm{L}}f(u)du\right|\le\frac{1}{2\mathrm{L}}\int_{\infty}^{\infty}|f(u)|du$$

當 $\mathrm{L}\to\infty$，它趨近於 0，因此，固定x，讓 $\mathrm{L}\to\infty$，(3.2)式變為

$$f(x)=\lim_{\mathrm{L}\to\infty}\frac{1}{\mathrm{L}}\sum_{n=1}^{\infty}\int_{-\mathrm{L}}^{\mathrm{L}}f(u)\cos\frac{n\pi(u-x)}{\mathrm{L}}du \tag{3.3}$$

令$\omega_n=\frac{n\pi}{\mathrm{L}}$，$\Delta\omega=\omega_{n+1}-\omega_n=\frac{\pi}{\mathrm{L}}$，(3.3)式變成

$$f(x)=\frac{1}{\pi}\lim_{\mathrm{L}\to\infty}\Delta\omega\sum_{n=1}^{\infty}\int_{-\mathrm{L}}^{\mathrm{L}}f(u)\cos\omega_n(u-x)du=\frac{1}{\pi}\int_0^{\infty}d\omega\int_{-\infty}^{\infty}f(u)\cos\omega(u-x)du$$

$$f(x)=\frac{1}{\pi}\int_0^\infty\int_{-\infty}^\infty f(u)\cos\omega(u-x)dud\omega \tag{3.4}$$

$$f(x)=\frac{1}{\pi}\int_0^\infty\int_{-\infty}^\infty [f(u)\cos\omega u\cos\omega x+f(u)\sin\omega u\sin\omega x]dud\omega$$

我們將上式寫成

$$f(x)=\int_0^\infty [a(\omega)\cos\omega x+b(\omega)\sin\omega x]d\omega \tag{3.5}$$

其中

$$a(\omega)=\frac{1}{\pi}\int_{-\infty}^\infty f(x)\cos\omega xdx$$

$$b(\omega)=\frac{1}{\pi}\int_{-\infty}^\infty f(x)\sin\omega xdx$$

(3.5)式為$f(x)$的 Fourier 積分表示式。

(1)若$f(x)$為偶函數，$b(\omega)=0$，(3.5)式化成

$$\begin{aligned} f(x)&=\int_0^\infty a(\omega)\cos\omega xd\omega \\ a(\omega)&=\frac{2}{\pi}\int_0^\infty f(x)\cos\omega xdx \end{aligned} \tag{3.6}$$

(3.6)式稱為$f(x)$的 Fourier 餘弦積分(Fourier cosine integral)

(2)若$f(x)$為奇函數，$a(\omega)=0$，(3.5)式化成

$$\begin{aligned} f(x)&=\int_0^\infty b(\omega)\sin\omega xd\omega \\ b(\omega)&=\frac{2}{\pi}\int_0^\infty f(x)\sin\omega xdx \end{aligned} \tag{3.7}$$

(3.7)式稱為$f(x)$的 Fourier 正弦積分(Fourier sine integral)

(3)因為 $\cos\omega(u-x)=\frac{1}{2}[e^{i\omega(u-x)}+e^{-i\omega(u-x)}]$，(3.4)式可寫成

$$f(x)=\frac{1}{2\pi}\int_0^{\infty}\int_{-\infty}^{\infty}f(u)[e^{i\omega(u-x)}+e^{-i\omega(u-x)}]dud\omega$$

$$=\frac{1}{2\pi}\int_0^{\infty}\int_{-\infty}^{\infty}f(u)e^{i\omega(u-x)}dud\omega+\frac{1}{2\pi}\int_0^{\infty}\int_{-\infty}^{\infty}f(u)e^{-i\omega(u-x)}dud\omega$$

$$=\frac{1}{2\pi}\int_0^{\infty}\int_{-\infty}^{\infty}f(u)e^{i(-\omega)(u-x)}dud(-\omega)+\frac{1}{2\pi}\int_0^{\infty}\int_{-\infty}^{\infty}f(u)e^{-i\omega(u-x)}dud\omega$$

$$=\frac{1}{2\pi}\int_{-\infty}^{0}\int_{-\infty}^{\infty}f(u)e^{-i\omega(u-x)}dud\omega+\frac{1}{2\pi}\int_0^{\infty}\int_{-\infty}^{\infty}f(u)e^{-i\omega(u-x)}dud\omega$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(u)e^{-i\omega(u-x)}dud\omega=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}f(u)e^{-i\omega u}du\right]e^{i\omega x}d\omega$$

我們可以寫成

$$f(x)=\int_{-\infty}^{\infty}c(\omega)e^{i\omega x}d\omega \tag{3.8}$$

其中

$$c(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}f(x)e^{-i\omega x}dx$$

(3.8)式稱為$f(x)$的複數型 Fourier 積分(complex form of the Fourier integral)。

[Fourier 積分定理]若$\int_{-\infty}^{\infty}|f(x)|\,dx$存在，$f(x)$分段平滑，則

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(u)e^{-i\omega(u-x)}dud\omega=\frac{f(x^-)+f(x^+)}{2}$$

Ex. 1

Find the Fourier integral of $f(x)=\begin{cases}1, & |x|<1 \\ 0, & |x|\geq 1\end{cases}$. [97 宜蘭電子 3]

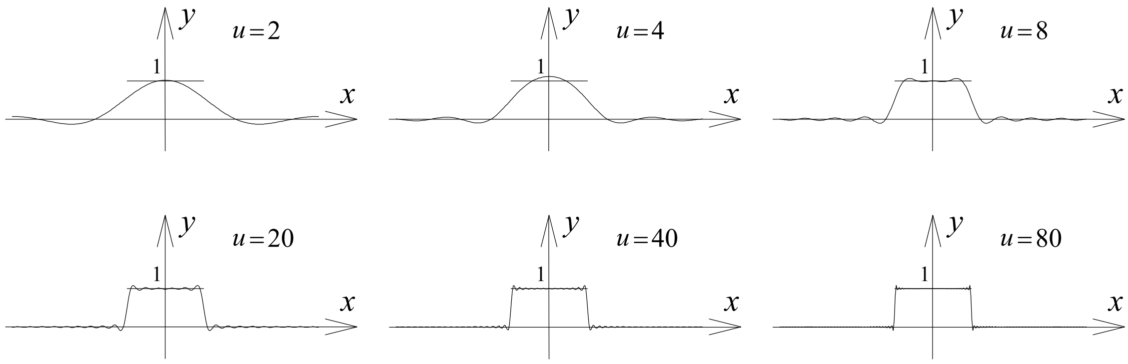
[解]令 $f(x)=\int_0^\infty [a(\omega)\cos\omega x+b(\omega)\sin\omega x]d\omega$

$$a(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos\omega x dx=\frac{1}{\pi}\int_{-1}^{1}\cos\omega x dx=\frac{1}{\pi\omega}[\sin(\omega)-\sin(-\omega)]=\frac{2\sin\omega}{\pi\omega}$$

$$b(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin\omega x dx=0$$

$$f(x)=\frac{2}{\pi}\int_0^\infty \frac{\sin\omega}{\omega}\cos\omega x d\omega$$

$\frac{2}{\pi}\int_0^u \frac{\cos\omega x\sin\omega}{\omega}d\omega$ 的圖形如下



Ex. 2

Let $f(x)=e^{-|x|}$, compute the complex Fourier integral of $f(x)$. Note that $f(x)=\begin{cases}e^{-x}, & x\geq 0 \\ e^{x}, & x<0\end{cases}$. [88 成大機械 5]

[解]令 $f(x)=\int_{-\infty}^{\infty} c(\omega)e^{i\omega x}d\omega$

$$c(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-|x|}e^{-i\omega x}dx=\frac{1}{2\pi}\left(\int_{-\infty}^{0} e^{x}e^{-i\omega x}dx+\int_0^\infty e^{-x}e^{-i\omega x}dx\right)$$

$$=\frac{1}{2\pi}\left(\int_{-\infty}^{0} e^{(1-i\omega)x}dx+\int_0^\infty e^{-(1+i\omega)x}dx\right)=\frac{1}{2\pi}\left(\left.\frac{e^{(1-i\omega)x}}{1-i\omega}\right|_{-\infty}^{0}-\left.\frac{e^{-(1+i\omega)x}}{1+i\omega}\right|_{0}^{\infty}\right)$$

$$=\frac{1}{2\pi}\left(\frac{1-0}{1-i\omega}-\frac{0-1}{1+i\omega}\right)=\frac{1}{2\pi}\cdot\frac{(1+i\omega)+(1-i\omega)}{(1-i\omega)(1+i\omega)}=\frac{1}{\pi(1+\omega^2)}$$

$$f(x)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{1+\omega^2}e^{i\omega x}d\omega$$

Ex. 3

Please use Fourier integral representation to show that $\int_0^\infty \frac{\cos x\omega + \omega \sin x\omega}{1+\omega^2} d\omega = \begin{cases} 0, & x<0 \\ \frac{\pi}{2}, & x=0 \\ \pi e^{-x}, & x>0 \end{cases}$.

[94 南大系統 3]

[解]令 $f(x) = \begin{cases} 0, & x<0 \\ e^{-x}, & x>0 \end{cases} \Rightarrow$ 設 $f(x) = \int_{-\infty}^{\infty} [a(\omega)\cos\omega x + b(\omega)\sin\omega x] d\omega$

先推導 $\int e^{ax} e^{ibx} dx = \int e^{(a+ib)x} dx = \frac{e^{(a+ib)x}}{a+ib} = \frac{(a-ib)e^{(a+ib)x}}{a^2+b^2} = \frac{e^{ax}(a-ib)(\cos bx + i\sin bx)}{a^2+b^2}$

$= \frac{e^{ax}[(a\cos bx + b\sin bx) + i(a\sin bx - b\cos bx)]}{a^2+b^2}$

實部 $\int e^{ax}\cos bx dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}$, 虛部 $\int e^{ax}\sin bx dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2}$

$a(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos\omega x dx = \frac{1}{\pi}\int_0^{\infty} e^{-x}\cos\omega x dx = \frac{1}{\pi}\cdot \left.\frac{e^{-x}(-\cos\omega x + \omega\sin\omega x)}{1+\omega^2}\right|_0^{\infty}$

$= \frac{0+1}{\pi(1+\omega^2)} = \frac{1}{\pi(1+\omega^2)}$

$b(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin\omega x dx = \frac{1}{\pi}\int_0^{\infty} e^{-x}\sin\omega x dx = \frac{1}{\pi}\cdot \left.\frac{e^{-x}(-\sin\omega x - \omega\cos\omega x)}{1+\omega^2}\right|_0^{\infty}$

$= \frac{0+\omega}{\pi(1+\omega^2)} = \frac{\omega}{\pi(1+\omega^2)}$

$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\cos\omega x + \omega\sin\omega x}{1+\omega^2} d\omega$

當 $x<0$ 時，$\frac{f(x^-)+f(x^+)}{2} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\cos\omega x + \omega\sin\omega x}{1+\omega^2} d\omega \Rightarrow \frac{0+0}{2} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\cos\omega x + \omega\sin\omega x}{1+\omega^2} d\omega$

$\int_{-\infty}^{\infty} \frac{\cos\omega x + \omega\sin\omega x}{1+\omega^2} d\omega = 0$

當 $x=0$ 時，$\frac{f(0^-)+f(0^+)}{2} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\cos\omega x + \omega\sin\omega x}{1+\omega^2} d\omega \Rightarrow \frac{0+1}{2} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\cos\omega x + \omega\sin\omega x}{1+\omega^2} d\omega$

$\int_{-\infty}^{\infty} \frac{\cos\omega x + \omega\sin\omega x}{1+\omega^2} d\omega = \frac{\pi}{2}$

當 $x>0$ 時，$\frac{f(x^-)+f(x^+)}{2} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\cos\omega x + \omega\sin\omega x}{1+\omega^2} d\omega$

$\frac{e^{-x}+e^{-x}}{2} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\cos\omega x + \omega\sin\omega x}{1+\omega^2} d\omega \Rightarrow \int_{-\infty}^{\infty} \frac{\cos\omega x + \omega\sin\omega x}{1+\omega^2} d\omega = \pi e^{-x}$

Ex. 4

Find the Fourier cosine integral for the function $f(t)=\begin{cases}2t, & 0<t<1\\ 0, & \text{otherwise}\end{cases}$. [102 虎尾車輛 2]

[解]設 $f(t)=\int_0^\infty a(\omega)\cos\omega t d\omega$

$$a(\omega)=\frac{2}{\pi}\int_0^\infty f(t)\cos\omega t dt=\frac{2}{\pi}\int_0^1 2t\cos\omega t dt=\frac{4}{\pi\omega}(t\sin\omega t\Big|_0^1-\int_0^1\sin\omega t dt)$$

$$=\frac{4}{\pi\omega}(\sin\omega+\frac{\cos\omega t}{\omega}\Big|_0^1)=\frac{4}{\pi\omega}(\sin\omega+\frac{\cos\omega-1}{\omega})$$

$$f(t)=\frac{4}{\pi}\int_{-\infty}^\infty(\frac{\sin\omega}{\omega}+\frac{\cos\omega-1}{\omega^2})\cos\omega t d\omega$$

[Exercises] 1. 求 $f(x)=\begin{cases}1, & -1<x<1\\ 0, & x<-1 \text{或} x>1\end{cases}$ 之傅立葉積分式。[104 高第一機械 9]

2. Find the Fourier integral representation of the following non-periodic function:

$$f(\theta)=\begin{cases}\cos\theta, & -\frac{\pi}{2}<x<\frac{\pi}{2}\\ 0, & \text{otherwise}\end{cases}$$. [104 高師大電子 5]

3. Please use the Fourier integral to show that

$$\int_0^\infty\frac{\sin\pi\omega\sin x\omega}{1-\omega^2}d\omega=\begin{cases}\frac{\pi}{2}\sin x, & \text{if } 0\le x\le\pi\\ 0, & \text{if } x>\pi\end{cases}$$. [101 暨南電機 5]

[Ans.]

1. $f(x)=\frac{2}{\pi}\int_0^\infty\frac{\sin\omega}{\omega}\cos\omega x d\omega$　　2. $f(\theta)=\frac{2}{\pi}\int_0^\infty\frac{1}{1-\omega^2}\cos\frac{\omega\pi}{2}\cos\omega\theta d\omega$

3. 略

II. 傅立葉轉換(Fourier Transform)

1. 傅立葉轉換的定義

由(3.8)式，若定義

$$\mathscr{F}[f(x)] \equiv F(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$$

則

$$\mathscr{F}^{1}[F(\omega)] = f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{F}(\omega)\, e^{i\omega x} d\omega$$

$F(\omega)$稱為$f(x)$的 Fourier 轉換。

同理，由(3.6)式，若

$$F_C(\omega) = \int_0^{\infty} f(x)\cos\omega x dx$$

則

$$f(x) = \frac{2}{\pi}\int_0^{\infty} F_C(\omega)\cos\omega x d\omega$$

$F_C(\omega)$稱為$f(x)$的傅立葉餘弦轉換(Fourier cosine transform)。

由(3.7)式，若

$$F_S(\omega) = \int_0^{\infty} f(x)\sin\omega x dx$$

則

$$f(x) = \frac{2}{\pi}\int_0^{\infty} F_S(\omega)\sin\omega x d\omega$$

$F_s(\omega)$稱為$f(x)$的傅立葉正弦轉換(Fourier sine transform)。

Ex. 5

Find the Fourier transform of the function $g(t) = \begin{cases} 2, & -3 < t < 1 \\ 0, & \text{otherwise} \end{cases}$. [105 南大電機 7]

[解]$\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt = \int_{-3}^{1} 2e^{-i\omega t}dt = \frac{2}{-i\omega}\cdot e^{-i\omega t}\Big|_{-3}^{1} = \frac{2}{-i\omega}(e^{-i\omega} - e^{i3\omega})$

Ex. 6

Find the Fourier transform of e^{-ax^2}, where $a>0$. [89 成大機械 5]

[解]$\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx = \int_{-\infty}^{\infty} e^{-ax^2}e^{-i\omega x}dx = \int_{-\infty}^{\infty} e^{-a(x^2+\frac{i\omega}{a}x)}dx = \int_{-\infty}^{\infty} e^{-a(x+\frac{i\omega}{2a})^2-\frac{\omega^2}{4a}}dx = e^{-\frac{\omega^2}{4a}}\int_{-\infty}^{\infty} e^{-a(x+\frac{i\omega}{2a})^2}dx$

令$u = x+\frac{i\omega}{2a}$，上式為

$$e^{-\frac{\omega^2}{4a}}\int_{-\infty}^{\infty} e^{-au^2}du = e^{-\frac{\omega^2}{4a}}\int_{-\infty}^{\infty} e^{-(\sqrt{a}u)^2}du = e^{-\frac{\omega^2}{4a}}\cdot\frac{1}{\sqrt{a}}\int_{-\infty}^{\infty} e^{-(\sqrt{a}u)^2}d(\sqrt{a}u) = e^{-\frac{\omega^2}{4a}}\cdot\sqrt{\frac{\pi}{a}}$$

Ex. 7

Determine the Fourier transform of $x(t)=\begin{cases}1+\cos\pi t, & |t|\le 1\\ 0, & |t|>1\end{cases}$. [106 台大海洋丙 2]

[解]設$x(t)=\int_{-\infty}^{\infty} c(\omega)e^{i\omega t}d\omega$

先推導$\int e^{ax}e^{ibx}dx = \int e^{(a+ib)x}dx = \frac{e^{(a+ib)x}}{a+ib} = \frac{(a-ib)e^{(a+ib)x}}{a^2+b^2} = \frac{e^{ax}(a-ib)(\cos bx + i\sin bx)}{a^2+b^2}$

$$= \frac{e^{ax}[(a\cos bx + b\sin bx) + i(a\sin bx - b\cos bx)]}{a^2+b^2}$$

實部$\int e^{ax}\cos bx dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}$, 虛部$\int e^{ax}\sin bx dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2}$

$$c(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} x(t)e^{-i\omega t}dt = \frac{1}{2\pi}\int_{-1}^{1}(1+\cos\pi t)e^{-i\omega t}dt = \frac{1}{2\pi}(\int_{-1}^{1} e^{-i\omega t}dt + \int_{-1}^{1}\cos\pi t e^{-i\omega t}dt)$$

$$= \frac{1}{2\pi}[\frac{e^{-i\omega t}}{-i\omega}\Big|_{-1}^{1} + \frac{e^{-i\omega t}(-i\omega\cos\pi t + \pi\sin\pi t)}{(-i\omega)^2+\pi^2}\Big|_{-1}^{1}]$$

$$= \frac{1}{2\pi}[\frac{e^{-i\omega}-e^{i\omega}}{-i\omega} + \frac{e^{-i\omega}(i\omega)-e^{i\omega}(i\omega)}{\pi^2-\omega^2}] = \frac{1}{2\pi}[\frac{-2i\sin\omega}{-i\omega} + \frac{i\omega(-2i\sin\omega)}{\pi^2-\omega^2}]$$

$$= \frac{1}{\pi}(\frac{\sin\omega}{\omega} + \frac{\omega\sin\omega}{\pi^2-\omega^2})$$

$$x(t) = \frac{1}{\pi}\int_{-\infty}^{\infty}(\frac{\sin\omega}{\omega} + \frac{\omega\sin\omega}{\pi^2-\omega^2})e^{i\omega t}d\omega$$

Ex. 8

Transform the function $f(x)=\begin{cases} x, & \text{if } 0<x<a \\ 0, & \text{if } x>a \end{cases}$ in the form of sine integral. [105 中正光電 6]

[解] $F_S(\omega)=\int_0^\infty f(x)\sin\omega x dx=\int_0^a x\sin\omega x dx=-\frac{1}{\omega}(x\cos\omega x\big|_0^a-\int_0^a\cos\omega x dx)$

$$=-\frac{1}{\omega}(a\cos\omega a-\frac{\sin\omega x}{\omega}\bigg|_0^a)=-\frac{1}{\omega}(a\cos\omega a-\frac{\sin\omega a}{\omega})=\frac{\sin\omega a-\omega a\cos\omega a}{\omega^2}$$

$$f(x)=\frac{2}{\pi}\int_0^\infty\frac{\sin\omega a-\omega a\cos\omega a}{\omega^2}\sin\omega x d\omega$$

[Exercises]1. Let a and k be positive numbers, and let $f(t)=\begin{cases} k, & -a\le t<a \\ 0, & t<-a, t\ge a \end{cases}$. Find the Fourier transform of $f(t)$. [104 南大電機 4]

2. 求下列函數的傅立葉轉換(a) $f(x)=\begin{cases} 1, & |x|<a \\ 0, & |x|>a \end{cases}$，其中 a 為常數；(b) $f(x)=\begin{cases} e^{-2x}, & x>0 \\ e^{5x}, & x<0 \end{cases}$ [102 虎尾光電 3]

3. Find the Fourier transform of the function $f(x)=xe^{-x^2}$. [105 中山光電 5、101 暨南電機 6]

4. Find the Fourier transform of the given function $f(x)=\begin{cases} xe^{-x}, & x>0 \\ 0, & x<0 \end{cases}$. [100 中原機械丙 5]

5. Show that the Fourier transformation of $e^{-\alpha t^2}$ is $\sqrt{\frac{\pi}{\alpha}}e^{-\frac{\omega^2}{4\alpha}}$. [106 台大機械 5(a)]

6. Find the Fourier transform of the function $f(x)=\begin{cases} x, & \text{if } 0<x<a \\ 0, & \text{otherwise} \end{cases}$. [105 彰師光電 3]

7. Find the Fourier cosine transformation of $f(x)=e^{-x}$. [103 高海電訊 4]

[Ans.]1. $\frac{2k\sin a\omega}{\omega}$ 2. (a) $\frac{2\sin a\omega}{\omega}$ (b) $\frac{70+7\omega^2-21i\omega}{(25+\omega^2)(4+\omega^2)}$ 3. $-\frac{i\omega}{2}\sqrt{\pi}e^{-\frac{\omega^2}{4}}$

4. $\frac{1}{(1+i\omega)^2}$ 5.略 6. $\frac{i\omega a e^{-i\omega a}+e^{-i\omega a}-1}{\omega^2}$ 7. $\frac{1}{1+\omega^2}$

III. Fourier 轉換的性質

令

$$F(\omega)=\mathscr{F}[f(t)]=\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$$

且

$$f(t)=\mathscr{F}^{-1}[F(\omega)]=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega$$

$$F_1(\omega)=\mathscr{F}[f_1(t)],\ F_2(\omega)=\mathscr{F}[f_2(t)]$$

$f(t)*g(t)=\int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau=\int_{-\infty}^{\infty} f(t-\tau)g(\tau)d\tau$ 定義為 $f(t)$ 及 $g(t)$ 的摺積

Fourier 轉換的性質

定理	原函數	轉換函數
線性	$c_1f_1(t)+c_2f_2(t)$	$c_1F_1(\omega)+c_2F_2(\omega)$
尺度改變	$f(at)$	$\frac{1}{\lvert a\rvert}F(\frac{\omega}{a})$
對稱	$F(t)$	$2\pi f(-\omega)$
共軛	$f^*(t)$	$F^*(-\omega)$
平移	$f(t-t_0)$	$F(\omega)\,e^{-i\omega t_0}$
	$f(t)\,e^{i\omega_0 t}$	$F(\omega-\omega_0)$
微分	$\frac{d^n f(t)}{dt^n}$	$(i\omega)^n F(\omega)$
	$(-it)^n f(t)$	$\frac{d^n F(\omega)}{d\omega^n}$
摺積	$f_1(t)*f_2(t)$	$F_1(\omega)F_2(\omega)$
	$f_1(t)f_2(t)$	$\frac{1}{2\pi}F_1(\omega)*F_2(\omega)$
Parseval	$\int_{-\infty}^{\infty}\lvert f(t)\rvert^2\,dt=\frac{1}{2\pi}\int_{-\infty}^{\infty}\lvert F(\omega)\rvert^2\,d\omega$	

1. 線性

$$\mathscr{F}[c_1f_1(t)+c_2f_2(t)]=\int_{-\infty}^{\infty}[c_1f_1(t)+c_2f_2(t)]e^{-i\omega t}dt=c_1\int_{-\infty}^{\infty}f_1(t)e^{-i\omega t}dt+c_2\int_{-\infty}^{\infty}f_2(t)e^{-i\omega t}dt$$

$$=c_1\mathscr{F}[f_1(t)]+c_2\mathscr{F}[f_2(t)]=c_1F_1(\omega)+c_2F_2(\omega)$$

2. 尺度改變

$\mathscr{F}[f(at)]=\int_{-\infty}^{\infty}f(at)e^{-i\omega t}dt$, Let $at=u$, we have

$$\mathscr{F}[f(at)]=\int_{-\infty}^{\infty}f(u)e^{-i\omega\frac{u}{a}}d\frac{u}{a}=\frac{1}{|a|}\int_{-\infty}^{\infty}f(u)e^{-i\frac{\omega}{a}u}du=\frac{1}{|a|}F(\frac{\omega}{a})$$

3. 對稱

因為

$$f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(\omega)e^{i\omega t}d\omega=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(\omega)e^{-i\omega(-t)}d\omega$$

將 t 與ω互換，得

$$2\pi f(-\omega)=\int_{-\infty}^{\infty}F(t)e^{-it\omega}dt=\int_{-\infty}^{\infty}F(t)e^{-i\omega t}dt$$

則 $2\pi f(-\omega)$是 $F(t)$的 Fourier 轉換，即

$$\mathscr{F}[F(t)]=2\pi f(-\omega)$$

4. 共軛

假設時間函數是複數，亦即$f(t)=x(t)+iy(t)$，它的共軛複數為$f^*(t)=x(t)-iy(t)$，則

$$\mathscr{F}[f(t)]=\int_{-\infty}^{\infty}[x(t)+iy(t)]e^{-i\omega t}dt=F(\omega)$$

且

$$\mathscr{F}[f^*(t)]=\int_{-\infty}^{\infty}[x(t)-iy(t)]e^{-i\omega t}dt=\left[\int_{-\infty}^{\infty}[x(t)+iy(t)]e^{i\omega t}dt\right]^*=[F(-\omega)]^*=F^*(-\omega)$$

若$f(t)$是實數時，則

$$f^*(t)=f(t)\Rightarrow\mathscr{F}[f^*(t)]=\mathscr{F}[f(t)]$$

$$F^*(-\omega)=F(\omega)\Rightarrow F(-\omega)=F^*(\omega)$$

5. 平移

(1)時間軸的平移

$$\mathscr{F}[f(t-a)]=\int_{-\infty}^{\infty}f(t-a)e^{-i\omega t}dt$$

令 $u=t-a$，右手邊為

$$\int_{-\infty}^{\infty}f(u)e^{-i\omega(u+a)}du=e^{-i\omega a}\int_{-\infty}^{\infty}f(u)e^{-i\omega u}du=e^{-i\omega a}F[f(t)]=F(\omega)e^{-i\omega a}$$

$$\Rightarrow\mathscr{F}[f(t-a)]=F(\omega)e^{-ia\omega}$$

(2)頻率軸的平移

$$\mathscr{F}^1[F(\omega-\omega_0)]=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(\omega-\omega_0)e^{i\omega t}d\omega=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(\xi)e^{i(\xi+\omega_0)t}d\xi$$
$$=\frac{1}{2\pi}e^{i\omega_0 t}\int_{-\infty}^{\infty}F(\xi)e^{i\xi t}d\xi=f(t)\,e^{i\omega_0 t}$$

6. 微分

(1)時間微分

$$\mathscr{F}[f'(t)] = \int_{-\infty}^{\infty} f'(t)e^{-i\omega t}dt = f(t)e^{-i\omega t}\Big|_{-\infty}^{\infty} - (-i\omega)\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt = (i\omega)\mathscr{F}[f(t)] = (i\omega)F(\omega)$$

同理

$$\mathscr{F}[f^{(n)}(t)] = (i\omega)^n F(\omega)$$

(2)頻率微分

$$\frac{d}{d\omega}F(\omega) = \frac{d}{d\omega}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt = \int_{-\infty}^{\infty}(-it)f(t)e^{-i\omega t}dt = \mathscr{F}[(-it)f(t)]$$

同理

$$\frac{d^n}{d\omega^n}F(\omega) = \mathscr{F}[(-it)^n f(t)]$$

7. 摺積

(1)時間摺積

$$\mathscr{F}[f_1(t)*f_2(t)] = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f_1(\tau)f_2(t-\tau)d\tau\right]e^{-i\omega t}dt = \int_{-\infty}^{\infty} f_1(\tau)\left[\int_{-\infty}^{\infty} f_2(t-\tau)e^{-i\omega t}dt\right]d\tau$$

由時間平移定理，中括號等於 $F_2(\omega)e^{-i\omega\tau}$，因此

$$\mathscr{F}[f_1(t)*f_2(t)] = \int_{-\infty}^{\infty} f_1(\tau)e^{-i\omega\tau}F_2(\omega)d\tau = F_1(\omega)F_2(\omega)$$

(2)頻率摺積

$$f_1(t)f_2(t)=\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}F_1(u)e^{iut}du\right]\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}F_2(\omega)e^{i\omega t}d\omega\right]=\frac{1}{2\pi}\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F_1(u)F_2(\omega)e^{i(u+\omega)t}dud\omega$$

令 $u+\omega=v\Rightarrow\omega=v-u$，得

$$f_1(t)f_2(t)=\frac{1}{2\pi}\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}F_1(u)F_2(v-u)du\right]e^{ivt}dv=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{1}{2\pi}F_1(v)*F_2(v)\right]e^{ivt}dv$$

$$\mathscr{F}[f_1(t)f_2(t)]=\frac{1}{2\pi}F_1(\omega)*F_2(\omega)$$

8. Parseval 定理

由摺積定理及共軛定理得

$$\mathscr{F}[f(t)f^*(t)]=\frac{1}{2\pi}F(\omega)*F^*(-\omega)$$

$$\int_{-\infty}^{\infty}f(t)f^*(t)e^{-i\omega t}dt=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(u)*F^*(-(\omega-u))du$$

令 $\omega=0$，則

$$\int_{-\infty}^{\infty}|f(t)|^2dt=\frac{1}{2\pi}\int_{-\infty}^{\infty}|F(u)|^2du=\frac{1}{2\pi}\int_{-\infty}^{\infty}|F(\omega)|^2d\omega$$

IV. 重要的 Fourier 轉換

1. $f(t)=\begin{cases}k, & |t|<a\\ 0, & |t|>a\end{cases}\quad\Leftrightarrow\quad F(\omega)=\dfrac{2k\sin a\omega}{\omega}$

2. $f(t)=\begin{cases}k, & 0<t<a\\ 0, & t>a\end{cases}\quad\Leftrightarrow\quad F_C(\omega)=\dfrac{k\sin a\omega}{\omega},\quad F_S(\omega)=\dfrac{k(1-\cos a\omega)}{\omega}$

3. $f(t)=u(t)e^{-at},\ a>0\quad\Leftrightarrow\quad F(\omega)=\dfrac{1}{a+i\omega},\ F_C(\omega)=\dfrac{a}{a^2+\omega^2},\ F_S(\omega)=\dfrac{\omega}{a^2+\omega^2}$

Ex. 9

請計算 $G(\omega)=\dfrac{1}{4-i(2-\omega)}$ 之 Fourier 反轉換。[104 東海電機 4]

[解] $\mathscr{F}[u(t)e^{-4t}]=\dfrac{1}{4+i\omega}\Rightarrow\mathscr{F}[u(t)e^{-4t}e^{i2t}]=\dfrac{1}{4+i(\omega-2)}\Rightarrow\mathscr{F}^{-1}[\dfrac{1}{4-i(2-\omega)}]=u(t)e^{(-4+2i)t}$

Ex. 10

(*a*) Compute the convolution of $f(x)$, $g(x)$ when $f(x)=g(x)=\begin{cases}1, & -a\le x\le a\\ 0, & |x|>a\end{cases}$

(*b*) Use the convolution theorem $H(\lambda)=F(\lambda)*G(\lambda)$ and the concept of inverse Fourier transform to evaluate $\int_{-\infty}^{\infty}\left(\dfrac{\sin\lambda}{\lambda}\right)^2 d\lambda$

Solution: (*a*) $g(x-t)=\begin{cases}1, & -a\le x-t\le a\\ 0, & |x-t|>a\end{cases}=\begin{cases}1, & x-a\le t\le x+a\\ 0, & |x-t|>a\end{cases}$

$$
\begin{aligned}
f(x)*g(x)&=\int_{-\infty}^{\infty}f(t)g(x-t)dt=\int_{-a}^{a}g(x-t)dt=\int_{-a}^{a}u(t-x+a)-u(t-x-a)dt\\
&=\big[(t-x+a)u(t-x+a)-(t-x-a)u(t-x-a)\big]_{-a}^{a}\\
&=[(2a-x)\,u(2a-x)-(-x)\,u(-x)]-[(-x)\,u(-x)-(-2a-x)\,u(-2a-x)]\\
&=(2a-x)\,u(2a-x)+2x\,u(-x)-(2a+x)\,u(-2a-x)
\end{aligned}
$$

(*b*) $\because F(\omega)=\int_{-\infty}^{\infty}f(x)e^{-i\omega x}dx=\int_{-a}^{a}e^{-i\omega x}dx=\dfrac{e^{-i\omega a}-e^{i\omega a}}{-i\omega}=\dfrac{2\sin\omega a}{\omega}=G(\omega)$

$$f(x)*g(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(\omega)G(\omega)e^{i\omega x}dx$$

$$(2a-x)u(2a-x)+2xu(-x)-(2a+x)u(-2a-x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{2\sin\omega a}{\omega}\right)^2 e^{i\omega x}d\omega$$

Let $x=0$, we get

$$2a=\frac{1}{2\pi}\times 4\int_{-\infty}^{\infty}\left(\frac{\sin\omega a}{\omega}\right)^2 d\omega=\frac{2a}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin\omega a}{\omega a}\right)^2 d(\omega a)$$

Let $\omega a=\lambda$, then

$$\frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin\lambda}{\lambda}\right)^2 d\lambda=2,\quad\Rightarrow\int_{-\infty}^{\infty}\left(\frac{\sin\lambda}{\lambda}\right)^2 d\lambda=\pi$$

Ex. 11

(a)Find the Fourier transform of $x(t)=\begin{cases}1, & |t|<1\\ 0, & |t|>1\end{cases}$. (b)Use (a) result to calculate $\int_{-\infty}^{\infty}\frac{\sin^2\omega}{\omega^2}d\omega$.

(Hint: Parseval's relation: $\int_{-\infty}^{\infty}|x(t)|^2\,dt=\frac{1}{2\pi}\int_{-\infty}^{\infty}|X(\omega)|^2\,d\omega$). [103 清大生醫甲 2]

[解](a) $X(\omega)=\int_{-\infty}^{\infty}x(t)e^{-i\omega t}dt=\int_{-1}^{1}e^{-i\omega t}dt=\frac{e^{-i\omega}-e^{i\omega}}{-i\omega}=\frac{2\sin\omega}{\omega}$

(b)由Parseval's關係式得

$$\int_{-\infty}^{\infty}|x(t)|^2\,dt=\frac{1}{2\pi}\int_{-\infty}^{\infty}|X(\omega)|^2\,d\omega \quad\Rightarrow\int_{-1}^{1}1^2\cdot dt=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{2\sin\omega}{\omega}\right)^2d\omega$$

$$2=\frac{2}{\pi}\int_{-\infty}^{\infty}\frac{\sin^2\omega}{\omega^2}d\omega \quad\Rightarrow\int_{-\infty}^{\infty}\frac{\sin^2\omega}{\omega^2}d\omega=\pi$$

[Exercises]1. $f(x)=\begin{cases}k, & -1\le x<1\\ 0, & \text{otherwise}\end{cases}$，(a)求 $f(x)$的傅立葉轉換(Fourier transform) (b)求 $\int_{-\infty}^{\infty}\frac{\sin^2\omega}{\omega^2}d\omega$. [101 虎尾電機 3]

2.

[Ans.]1. (a) $\frac{2k\sin\omega}{\omega}$ (b)π 2.